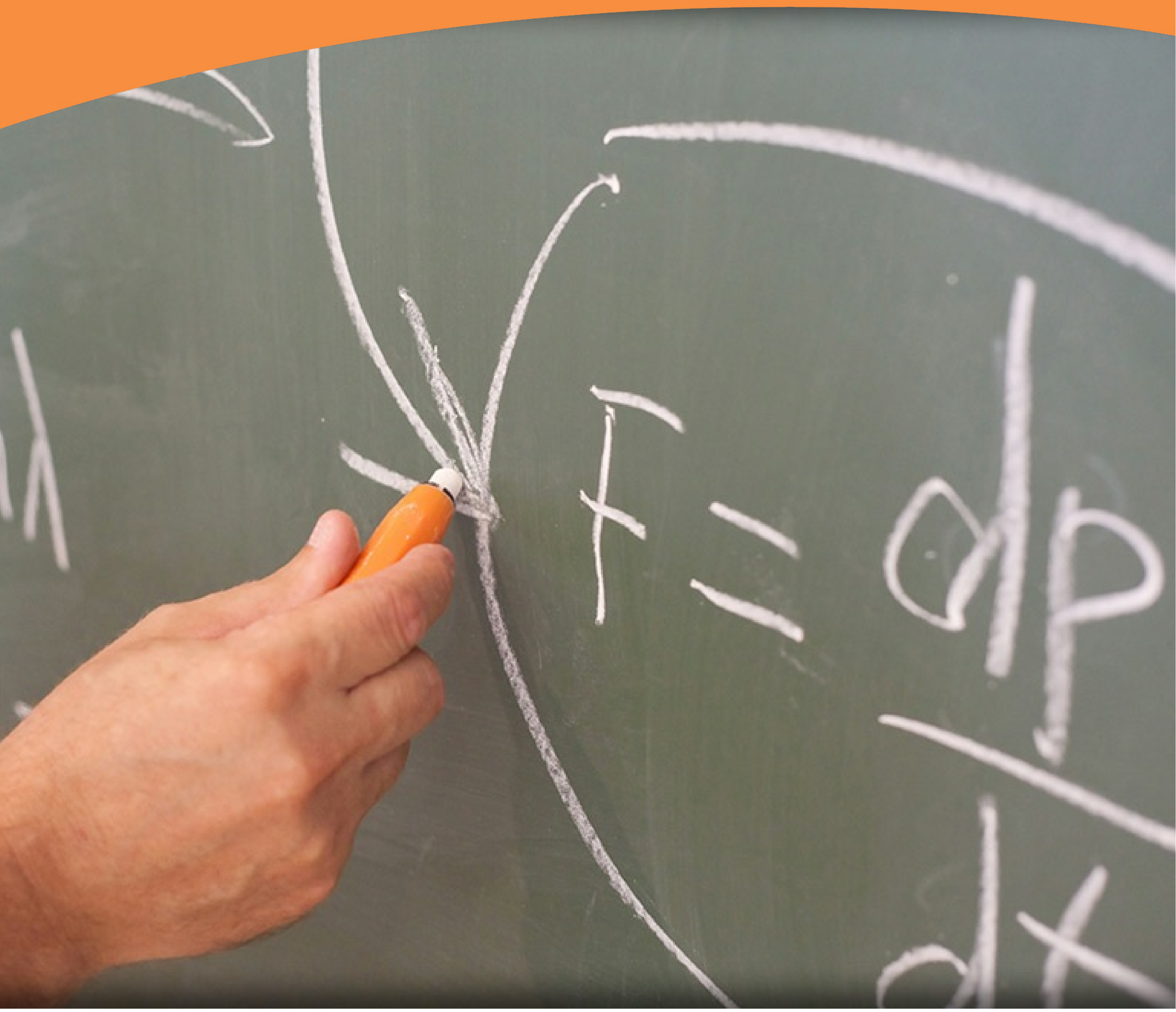
Mechanics, Thermodynamics, Oscillations and waves

College Physics I: Notes and exercises

Daniel Gebreselasie



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College Physics I: Notes and Exercises

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1 Introduction to Mechanics

Your goals for this chapter are to learn about measurement, significant figures, conversion of units, introduction to trigonometry and coordinate systems.

1.1 Measurement

Measurement is comparison with a standard. For example when we say the length of a certain object is *3* m we are saying the length of the object is *3* times the length of the standard meter. The standard with which the comparison is made is called a unit of measurement. For example the unit of measurement for length is the meter

There are two systems of units. These are the British System and the SI system. SI is an abbreviation for the French phrase 'Systeme Internationale'. The British system is used in the United States while the SI system is used in most of the rest of the world. For scientific purposes, the SI system is used. In this course, the SI system will be used.

1.2 The SI System of Units

The SI system of units are the units based on standards kept in an SI office in France. These units may be classified into two: Fundamental units and derived units.

1.2.1 Fundamental Units

These are the minimum set of units from which all the units of physics can be assembled. The standards for SI fundamental units are kept in the SI office in France. Manufacturers of fundamental units should base their units on these standards. The following table is a list of the fundamental units of physics.

Physical Quantity	Unit	Abbreviation
Length	meter	m
Time	second	s
Mass	kilogram	kg
Temperature	degree Kelvin	°K
Current	Ampere	A

Table 1.1

The units of length (meter), time (second) and mass (kilogram) are the fundamental unit of Mechanics. The unit of temperature (°K) is the fundamental unit of thermodynamics (study of heat). The unit of current (A) is the fundamental unit of electricity and magnetism.

1.2.2 Derived Units

Derived units are units that can be expressed as a combination of fundamental units. For example the unit of speed is a derived unit because it can be expressed as a ratio between the unit of length and the unit of time (m / s). The following table is a list of some derived units of physics.

Physical Quantity	Unit	Abbreviation
Speed	meter/second	m/s
Acceleration	meter/second^2	m/s^2
Volume	meter^3	m^3
Density	kilogram/meter^3	kg/m^3

Table 1.2

1.2.3 Common Abbreviations of Powers of Ten

The units of physics are defined in such a way that they can be comprehended by human senses. For example the kilogram is a weight we can hold in our hand; the second is an interval of time we can comprehend and the meter is about twice of human arm. But in physics, quite often we deal with quantities which are either much bigger or much smaller than these units. To deal with such quantities conveniently, names and abbreviations for some powers of ten are defined. For example the "kilo" and "k" are the name and the abbreviation for a *1000*. The following table is a list of some commonly used powers of ten.

Power of Ten	Name	Abbreviation
10^3	kilo	k
10^6	Mega	M
10^9	Gega	G
10^{-1}	deci	d
10^{-2}	centi	c
10^{-3}	milli	m
10^{-6}	micro	μ
10^{-9}	nano	n

Table 1.3

1.3 Significant Figures

Significant figures are digits of a report of a measurement that make sense. For example let say the length of a certain object is measured by a ruler whose least count is a cm and the measurement is reported as *4.321* cm. Using this device, it is possible to determine that the length of the object is between *4* cm and *5* cm. The ones digit (*4*) can be determined exactly. The tenth digit (*3*) can be determined approximately because we know the length is between *4* cm and *5* cm. Since we are not sure of the tenth digit (*3*), it is impossible to determine the hundredth digit (*2*) and the thousandth digit (*1*). Thus, we say the ones digit (*4*) and the tenth digit (*3*) are significant digits while the hundredth digit (*2*) and the thousandth digit (*1*) are not significant digits.

Scientifically, significant digits include all accurate digits and one uncertain digit. A report of a measurement should include only significant digits. For example the length of the object in our example should be reported as *4.3* cm. But sometimes we will be forced to include zeroes that are not significant digits because zeroes are used to hold decimal places. For example let say the length of an object is measured by a device whose least count is *100* cm and its length is found to be between *200* cm and *300* cm. The tens digit can be approximated (let say it is *5*) but the ones digit cannot be determined. Even though the ones digit cannot be determined we have to put zero in its place in order to indicate that the digit *2* is a hundreds digit. The measurement is reported as *250* cm. It is important that we are able to tell whether a zero included in a report of a measurement is significant or not

1.3.1 Determining whether Zeroes in a Report of Measurement are Significant or not.

We can determine whether a zero included in a report of a measurement is significant or not by following the following rules.

1. A non-zero digit in a report of a measurement is always significant.
2. Tailing zeroes before the decimal point are not significant. They are used to hold decimal places only. For example the zeroes in *200* are not significant.
3. Tailing zeroes after the decimal point are significant. If they are not significant, they don't have to be included. For example the zeroes in *2.1000* are significant.
4. Leading zeroes are not significant. They are used to hold decimal places only. For example the zeroes in *0.0023* are not significant.
5. Zeroes located between significant digits are significant. This is because the zero holds a higher decimal place than another significant digit. For example the zeroes in 3001 are significant.

Example: How many significant digits are there in the following reports of a measurement?

a) *3000*
   Solution: one, because the zeroes are tailing zeroes before the decimal point.

---

b) *2001*
Solution: four, because the zeroes are located between significant digits

---

c) *0.00453*
Solution: three, because the zeroes are leading zeroes

3.1000
Solution: five, because the zeroes are tailing zeroes after the decimal point.

---

d) *100.0*
Solution: four, because the last zero is a tailing zero after the decimal point and the other zeroes are located between significant digits

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1.3.2 Operating with Significant Figures

When operating (adding, subtracting, multiplying, dividing) with significant figures, the result cannot be more accurate (have greater number of significant digits) than either of the figures being operated.

1.3.3 Adding or Subtracting Significant Figures

The result of adding or subtracting significant figures should have the same number of decimal places as the figure with the least number of decimal places. For example when adding *2.13* and *3.4571*, the sum should have only two decimal places because one of the figure (*2.13*) has two decimal places (*1* and *3*) and the second number (*3.4571*) has four decimal places (*4, 5, 7* and 1). Even though the algebraic addition of the figures gives *5.5871*, to obtain significant figures this should be rounded to two decimal places and the result should be reported as *5.59*.

1.3.4 Multiplying or Dividing Significant Figures

The result of multiplying or dividing significant numbers should have the same number of significant digits as the figure with the least number of significant digits. For example when multiplying *200* by *38*, the result should have only one significant digit because one of the figures (*200*) has only one significant digit and the other figure (*38*) has two significant digits. Even though direct multiplication gives *7600*, to obtain significant figures, this should be rounded to ten thousands decimal place and the result should be reported as *8000*.

1.4 Standard (Scientific) Notation

Expressing a number in standard notation means expressing a number as a product between a number between one (inclusive) and ten and a power of ten. It allows you to express a report of a measurement as a product of a number that consists of significant digits only and a power of ten. The non-significant zeroes are absorbed in the power of ten. For example to express *2400* in standard notation, first we have to divide it by 1000 to get a number between one (inclusive) and ten. This gives *2.4*. And then of course we have to multiply by a *1000* or 10^3 to represent the original number (*2400*). Thus the standard notation of *2400* is 2.4×10^3. Similarly, to express *0.0540* in standard notation first we multiply it by *100* to change it to a number between one (inclusive) and ten which gives *5.40* and then multiply it by *10* to the power of *-2*. Thus, its standard notation is 5.40×10^{-2}. The zero is included in *5.40* because it is significant.

1.4.1 Adding or Subtracting Numbers in Standard Notation

To add or subtract numbers in standard notation, first manipulate the numbers so that all of them have the same powers of ten. Then, factor out the power of ten and operate. For example to add the numbers 2×10^2 and 3×10^3, first change the power of ten of the first number to *3* by dividing the *2* by *10* and multiplying the power of ten by *10*. This gives 0.2×10^3. Then factor out the power of ten to get $(0.2 + 3) \times 10^3$. And the result is 3.2×10^3.

1.4.2 Multiplying or Dividing Numbers in Standard Notation

To multiply or divide numbers in standard notation, multiply (divide) numbers with numbers and powers of ten with powers of ten. For example to multiply 2×10^2 and 3×10^3, multiply the numbers (*2* and *3*) together and the powers of ten (10^2 and 10^3) together to get 6×10^5.

The following list of laws of exponents may be useful in operating with numbers in standard notation:

$$x^a x^b = x^{a+b}$$

$$x^a / x^b = x^{a-b}$$

$$x^0 = 1$$

$$x^{-a} = 1/x^a$$

$$(x^a)^b = x^{ab}$$

$$x^a y^a = (xy)^a$$

$$x^a / y^a = (x/y)^a$$

1.5 Practice Quiz 1.1

Choose the best answer. Answers can be found at the back of the book.

1. The System of units used for scientific purposes is the (1 point)
   A. British system
   B. American system
   C. SI system
   D. European system

2. *2* cm means (1 point)
   A. *2e-2* m
   B. *2e2* m
   C. *2e-3* m
   D. *2e-1* m

3. Which of the following is a fundamental unit? (1 point)
   A. unit of time
   B. unit of volume
   C. unit of speed
   D. unit of force

4. The measurement of the length of a certain rod is reported as *12.34* cm. Which of these digit(s) is (are) uncertain? (1 point)
   A. *3* and *4*
   B. all of the digits are accurate
   C. *4*
   D. *2*, *3* and *4*

5. How many significant digits are there in the significant figure 0.00005643 (1 point)
   A. 9
   B. 1
   C. 8
   D. 4

6. How many significant digits are there in the significant figure 0.004000564 (1 point)
   A. 9
   B. 10
   C. 7
   D. 4

7. How many significant digits are there in the significant figure 2000.0 (1 point)
   A. 2
   B. 1
   C. 4
   D. 5

8. The number of significant digits in the significant figures 2400.00, 0.030, and 5030 respectively are (2 point)
   A. 2, 2, and 3
   B. 4, 1 and 4
   C. 2, 1, and 3
   D. 6, 2, and 3

9. The sum of the significant figures 9.02, 2.1, and 12.123 gives the significant figure (1 point)
   A. 23
   B. 23.4
   C. 23.243
   D. 23.2

10. The product of the significant figures 456, 300 and 12 gives the significant figure (1 point)
    A. 2000000
    B. 1600000
    C. 1000000
    D. 1641600

11. Express *560000* in standard notation (1 point)
    A. *56e4*
    B. *5.6e4*
    C. *5.6e5*
    D. *5.6e-5*

12. Express *0.000000786* in standard notation (1 point)
    A. *7.86e-8*
    B. *7.86e-6*
    C. *7.86e8*
    D. *7.86e-7*

1.6 Conversion of Units

To convert from one unit to another, first find a relationship between the units. Then use this relationship to convert from one to the other. A relationship between the units may be obtained by finding the ratio between the two units.

Example: Convert 5 km to mm.

Solution: First we have to find a relationship between km and mm by finding the ratio between km and mm. The unit meter (m) cancels out, so the ratio is basically ratio between kilo (k) and milli (m). Remember k = 10^{3} and m = 10^{-3}.

$$\mathrm{km}/\mathrm{mm} = \mathrm{k}/\mathrm{m} = 10^{3}/10^{-3} = 10^{6}$$

$$\mathrm{km} = 10^{6}\ \mathrm{mm}$$

$$5\ \mathrm{km} = 5 \times 10^{6}\ \mathrm{mm}$$

Example: Convert $5\ \text{cm}^2$ to m^2

Solution: First we have to find the ratio between the two units. The unit (m^2) will cancel out and the ratio simplifies to c^2 (The square in cm applies to both c and m). Remember $c = 10^{-2}$.

$$\text{cm}^2/\text{m}^2 = c^2 = 10^{-4}$$

$$\text{cm}^2 = 10^{-4}\ \text{m}^2$$

$$5\ \text{cm}^2 = 5 \times 10^{-4}\ \text{m}^2$$

1.7 Dimensional Analysis

Dimensional Analysis is a method used to determine if an equation is sound or not by comparing the units of both sides of the equation. Both sides of an equation are obviously expected to have the same units. If they turn out to be different, then the equation must be wrong; if they turn out to be the same, then the equation is at least dimensionally correct and there is a good chance it may be correct.

Example: Determine if the equation *speed = acceleration x time* is dimensionally correct.

Solution: The unit for speed is m/s. The unit for acceleration is m/s^2. Thus, the unit for the left side of the equation is m/s and that of the right hand side is the product of m/s^2 and s which is equal to m/s. Since both sides have the same units, the equation is dimensionally correct.

1.8 Order of Magnitude Calculation

Calculating the order of magnitude of a certain calculation means approximating its power of ten. To obtain the order of magnitude, replace every number by the power of ten closest to it and then carry out the calculation.

Example: Obtain the order of magnitude of the following calculation: $34 * 4 * 786 * 9876$

Solution: 34 is between 10^1 and 10^2 and is closer to 10^1. 4 is between 10^0 and 10^1 and is closer to the former. 786 is between 10^2 and 10^3 and is closer to the later. 9876 is between 10^3 and 10^4 and is closer to the later. Therefore the order of magnitude of this calculation is

$$10^1 * 10^0 * 10^3 * 10^4 = 10^8$$

1.9 Brief Review of Trigonometry

1.9.1 Trigonometric Functions

There are three basic trigonometric functions which are the cosine, the sine and the tangent. They are defined in terms of a right angled triangle. A right angled triangle is a triangle whose largest angle's measure is $90°$. Its longest side or the side opposite to the 90 degree angle is called the hypotenuse. The other two sides are called the legs of the right angled triangle. Now consider one of the none 90° angles. This angle is formed by the hypotenuse and one of the legs. The latter is called the adjacent side. The other leg which is not part of this angle is called the opposite side. Let this angle be x. the adjacent side be a, the opposite side be b and the hypotenuse be c.

The cosine of this angle, written as $\cos x$, is defined to be the ratio between the adjacent side and the hypotenuse.

$$\cos x = a/c$$

The sine of this angle, written as $\sin x$, is defined to be the ratio between the opposite side and the hypotenuse.

$$\sin x = b/c$$

The tangent of this angle, written as $tan\ x$, is defined to be the ratio between the opposite side and the opposite side.

$$tan\ x = b / a$$

The values of trigonometric functions are available in a scientific calculator.

The hypotenuse and the legs of a right angled triangle are related by Pythagorean Theorem.

$$c^2 = a^2 + b^2$$

Example: The degree measure of one of the angles of a right angled triangle is 60°. The hypotenuse is *10*. Calculate its adjacent side.

Solution: $c = 10$; $x = 60°$; $a =?$

The trigonometric function that relates these 3 values is cosine.

$$cos\ x = a/c$$

$$a = c\ cos\ x = 10\ cos\ 60° = 5$$

1.9.2 Inverse Trigonometric Functions

The inverses of trigonometric functions, called inverse trigonometric functions, help you to recover an angle from the value of a trigonometric function.

Cosine inverse of a trigonometric value *a*, written as $cos^{-1}\ a$ or *arccos a*, is defined as follows:

If

$$cos\ x = a,$$

then

$$x = arccos\ a$$

Sine inverse of a trigonometric value *a*, written as $sin^{-1}\ a$ or *arcsin a*, is defined as follows:

If

$$\sin x = a,$$

then

$$x = \arcsin a$$

Tangent inverse of a trigonometric value a, written as $\tan^{-1} a$ or $\arctan a$, is defined as follows:

If

$$\tan x = a,$$

then

$$x = \arctan a$$

Example: In a certain right angled triangle of hypotenuse *50*, the side opposite to one of the angles, *x*, is *25*. Calculate the angle.

Solution: $c = 50$; $b = 25$; $x = ?$

$$\sin x = b/c = 25/50 = 0.5$$

$$x = \arcsin 0.5 = 30°$$

1.9.3 Trigonometric Identities

The following is a list of some common trigonometric identities.

$$\tan x = (\sin x)/(\cos x)$$

$$\cos^2 x + \sin^2 x = 1$$

$$\sin (x \pm y) = \sin (x) \cos (y) \pm \cos (x) \sin (y)$$

$$\cos (x \pm y) = \cos (x) \cos (y) \pm (-) \sin (x) \sin (y)$$

$$\sin (2x) = 2 \sin x \cos x$$

$$\cos (2x) = \cos^2 x - \sin^2 x$$

1.10 Coordinate Systems

A coordinate system is a system for associating a set of three numbers with points in space uniquely. Two special cases of this are the one dimensional coordinate system and the two dimensional coordinate system.

1.10.1 One Dimensional Coordinate System

The one dimensional coordinate system is also called a number line. It is a system that associates single numbers with points on a line uniquely. A point is related with a number that is equal to its distance from a certain point that we call a reference point or origin. To distinguish between distances to the right of the origin and distances to the left of the origin, distances to the right of the origin are taken to be positive while distances to the left of the origin are taken to be negative.

1.10.2 Two Dimensional Coordinate System

A Two dimensional coordinate system is also called a coordinate plane. It is a system for associating pairs of numbers with points in a plane uniquely. There are two kinds of two dimensional coordinate system which are the Cartesian coordinate system and the polar coordinate system.

1.10.3 The Cartesian coordinate system

In this kind of coordinate system, a point is related with its perpendicular distances from two reference lines that are perpendicular to each other. The vertical reference line is called the y-axis and the horizontal reference line is called the x-axis (They don't have necessarily to be vertical and horizontal. But the horizontal-vertical coordinate system is the most common one). The intersection point of this two lines is called the origin. The perpendicular distance between the point and the y-axis is called the x-coordinate of the point. Distances to the right of the y-axis are taken to be positive while distances to the left of the y-axis are taken to be negative. The perpendicular distance between the point and the x-axis is called the y-coordinate of the point. The y-coordinates for points above the x-axis are taken to be positive and y-coordinates for points below the x-axis are taken to be negative. The x-coordinate and y-coordinate of a point are customarily represented by the letters x, and y respectively. The coordinate of the point is represented by the ordered pair (x, y).

1.10.4 The Polar Coordinate System

In this kind of coordinate system, a point is related to its distance from the origin and the angle formed between the line joining the origin to the point and the positive x-axis. The angle is taken to be positive if measured in a counter clockwise direction from the positive x-axis and negative if measured in a clockwise direction from the positive x-axis. The distance and the angle are customarily represented by r and θ respectively. The coordinate of the point is represented by the ordered pair (r, θ).

1.10.5 Relationship between Cartesian and Polar Coordinates

Consider the right angled triangle formed by the side joining the origin to the given point and a line extended from the point to the x-axis perpendicularly. The length of the hypotenuse is equal to r. The angle formed between the horizontal leg and the hypotenuse is equal to θ. The lengths of the horizontal and vertical legs are respectively equal to the x and y coordinates of the point (if in the first quadrant). The following relationships between Cartesian and polar coordinates can be obtained easily using the definitions of trigonometric functions.

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$r = \sqrt{(x^2 + y^2)}$$

$$\theta = \arctan (y/x)$$

Example: The polar coordinate of a certain point is (*100, 60°*). Calculate its Cartesian coordinates.

Solution: $r = 100$; $\theta = 60°$; $x = ?$; $y = ?$

$$x = r \cos \theta = 100 \cos 60° = 50$$

$$y = r \sin \theta = 100 \sin 60° = 87$$

Example: The Cartesian coordinate of a certain point is *(64, 48)* m. Calculate its polar coordinates. *Solution*: $x = 64$; $y = 36$; $r = ?$; $\theta = ?$

$$r = \sqrt{(x^2 + y^2)} = \sqrt{(64^2 + 48^2)} = 80$$

$$\theta = \arctan (y/x) = \arctan (48/64) = 37°$$

1.11 Practice Quiz 1.2

Choose the best answer. Answers can be found at the back of the book.

1. Use dimensional analysis to determine which of the following equations is dimensionally correct
   A. *acceleration * time*^{2} *= speed / time*
   B. *acceleration * time*^{2} *= distance / time*
   C. *acceleration * time*^{2} *= distance * time*
   D. *acceleration * time*^{2} *= speed * time*
   E. *acceleration * time*^{2} *= speed*^{2}

2. Approximate the order of magnitude of the following product: *346 * 8 * 5632 * 25 * 3 * 980*
   A. *1e9*
   B. *1e11*
   C. *1145819136000*
   D. *1e10*
   E. *1e13*

3. *8* micro m is equal to
   A. *8e-3* nm
   B. *8e3* nm
   C. *8e2* nm
   D. *8e-2* nm
   E. *8e6* nm

4. *7.2* km / hr is equal to
   A. *25.92* m / s
   B. *7200* m / s
   C. *2592* m / s
   D. *2000* m / s
   E. *2* m / s

5. *4* m^2 is equal to
   A. *4e2* cm^2
   B. *4e-4* cm^2
   C. *4e1* cm^2
   D. *4e-2* cm^2
   E. *4e4* cm^2

6. 1 km / hr ^{2} is equal to (1 point)
   A. $5e\text{-}5$ m / s ^{2}
   B. $7.716e\text{-}5$ m / s 2
   C. $6.48e4$ m / s ^{2}
   D. $7.716e\text{-}4$ m / s ^{2}
   E. $6.48e3$ m / s ^{2}

7. The hypotenuse of a right angled triangle is 13 m long. The length of one of the legs is 2. Calculate the length of the other leg.
   A. *10.276* m
   B. *12.845* m
   C. *7.707* m
   D. *6.423* m
   E. *8.992* m

8. The hypotenuse of a right angled is *70* m. If the angle between this leg and the hypotenuse is *80* degrees, calculate the length of one of the legs.
   A. *17.018* m
   B. *12.155* m
   C. *19.449* m
   D. *14.586* m
   E. *7.293* m

9. The hypotenuse of a right angled triangle is *70* m long. Calculate the length of one of its legs, if the angle opposite to this leg is *70* degrees.
   A. *72.356* m
   B. *52.623* m
   C. *65.778* m
   D. *78.934* m
   E. *85.512* m

10. One of the legs of a right angled triangle is *30* m long. The angle between this leg and the hypotenuse is *30* degrees. Calculate the length of the other leg.
    A. *20.785* m
    B. *10.392* m
    C. *13.856* m
    D. *19.053* m
    E. *17.321* m

11. The legs of a right angled triangle are respectively *11* m and *16* m. Calculate the angle formed between the former leg and the hypotenuse.
    A. *33.295* deg
    B. *38.844* deg
    C. *55.491* deg
    D. *49.942* deg
    E. *44.393* deg

12. The Cartesian coordinates of a certain point are (x, y) = (*55, 20*) m. Calculate the polar coordinates (r, θ).
    A. (*46.819* m, *9.992* deg)
    B. (*58.523* m, *21.981* deg)
    C. (*58.523* m, *19.983* deg)
    D. (*70.228* m, *19.983* deg)
    E. (*70.228* m, *21.981* deg)

13. The polar coordinates of a certain point are (r, θ) = (*22* m, *10* deg). Calculate its Cartesian coordinates (x, y).
    A. (*25.999* m, *4.202* m)
    B. (*21.666* m, *3.056* m)
    C. (*21.666* m, *3.82* m)
    D. (*19.499* m, *3.82* m)
    E. (*25.999* m, *5.348* m)

2 One Dimensional Motion

Your goals for this chapter are to learn about the variables used to describe motion in a straight line and how they are related to each other.

Motion is change in location with time. One dimensional motion is motion in a straight line. This kind of motion can be described by a single number or by a number line. It can be dealt with the simple algebra of numbers.

2.1 Variables of Motion

The variables used to describe motion are position, displacement, velocity and acceleration.

2.1.1 Position

Position (x) is a physical quantity used to represent the location of a particle. Position is specified with respect to a certain reference point that we call the origin. The value of the position of a certain particle is not absolute. It depends on the choice of reference point. It will have different values for different choices of reference points. The SI unit of measurement for position is the meter. Positions to the right of the reference point are taken to be positive while positions to the left of the reference point are taken to be negative.

Example: Consider a number line marked with the letters A through J with consecutive letters separated by 1 cm.

a) Draw the number line with its markings.

Solution:

Figure 2.1

b) Assuming the origin is fixed at point D, Obtain the positions of points B, D and G.

Solution: Point B is 2 cm to the left of point D.

$$x_B = -2 \text{ cm}$$

Point D is the origin itself.

$$x_D = 0 \text{ cm}$$

Point G is 3 cm to the right of point D.

$$x_G = 3 \text{ cm}$$

2.1.2 Displacement

Displacement (Δx) of a particle is defined to be change in the position of a particle.

$$\Delta x = x_f - x_i$$

Where x_i is initial position and x_f is final position. The SI unit of measurement for displacement is the meter. The displacement of a particle does not depend on the choice of a reference point. Displacement to the right is taken to be positive while displacement to the left is taken to be negative.

Example: Consider motion on a number line.

a) Calculate the displacement of a particle displaced from $x = 3$ cm to $x = 7$ cm.

Solution: $x_i = 3$ cm; $x_f = 7$ cm; $\Delta x = ?$

$$\Delta x = x_f - x_i = (7 - 3) \text{ cm} = 4 \text{ cm}$$

b) Calculate the displacement of a particle displaced from = 5 cm to $x = -7$ cm.

Solution: $x_i = 5$ cm; $x_f = -7$ cm; $\Delta x = ?$

$$\Delta x = x_f - x_i = (-7 - 5) \text{ cm} = -12 \text{ cm}.$$

c) A particle is displaced from $x = 3$ cm to $x = 10$ cm and then back to $x = 1$ cm.

i. Calculate the distance travelled.

Solution: Distance (d) is equal to the length of the path travelled.

$$d = |10 - 3| \text{ cm} + |1 - 10| \text{ cm} = 16 \text{ cm}.$$

ii. Calculate its displacement.

Solution: Displacement depends only on the initial and final position only: $x_i = 3$ cm; $x_f = 1$ cm;

$$\Delta x; = x_f - x_i = (1 - 3) \text{ cm} = -2 \text{ cm}.$$

2.1.3 Average Velocity

Average Velocity (v_{av}) is defined to be displacement per a unit time.

$$v_{av} = (x_f - x_i)/\Delta t$$

Where Δt is interval of time during which the displacement took place. The SI unit for average velocity is meter per second (m/s). Velocity to the right is taken to be positive and that to the left is taken to be negative. Average velocity between two events can be obtained from a graph of position versus time as the slope of the line joining the two events (points).

Example: Calculate the average velocity of a particle displaced from $x = 20$ cm to $x = 15$ cm in 10 seconds.

Solution: $x_f = 20$ cm; $x_f = 15$ cm; $\Delta t = 10$ s.

$$v_{av} = (x_f - x_i)/\Delta t = (15 - 20)/10 = -5 \text{ cm/s}.$$

Example: A particle is displaced from $x = 5$ cm to $x = 10$ cm in 7 seconds and then back to $x = 5$ cm in 3 seconds.

a) Calculate its average speed.

   Solution: Average speed is defined to be total distance (d) over total time (t): $d = 10$ cm; $t = 10$ s. *average speed* = ?

$$\textit{average speed} = d/t = 10/10 \text{ cm/s} = 1 \text{ cm/s}.$$

---

b) Calculate its average velocity.

Solution: $x_i = 5$ cm; $x_f = 5$ cm; $\Delta t = 10$ s.

$$v_{av} = (x_f - x_i)/\Delta t = (5 - 5)/10 \text{ cm/s} = 0 \text{ cm/s}.$$

2.1.4 Instantaneous Velocity

Instantaneous Velocity (v) is defined to be velocity at a given instant of time. In other words, it is average velocity evaluated at a very small interval of time. Instantaneous velocity at a given instant of time may be obtained from a graph of position versus time as the slope of the line tangent to the curve at the given point.

Example: The following is a graph of position versus time for a certain particle.

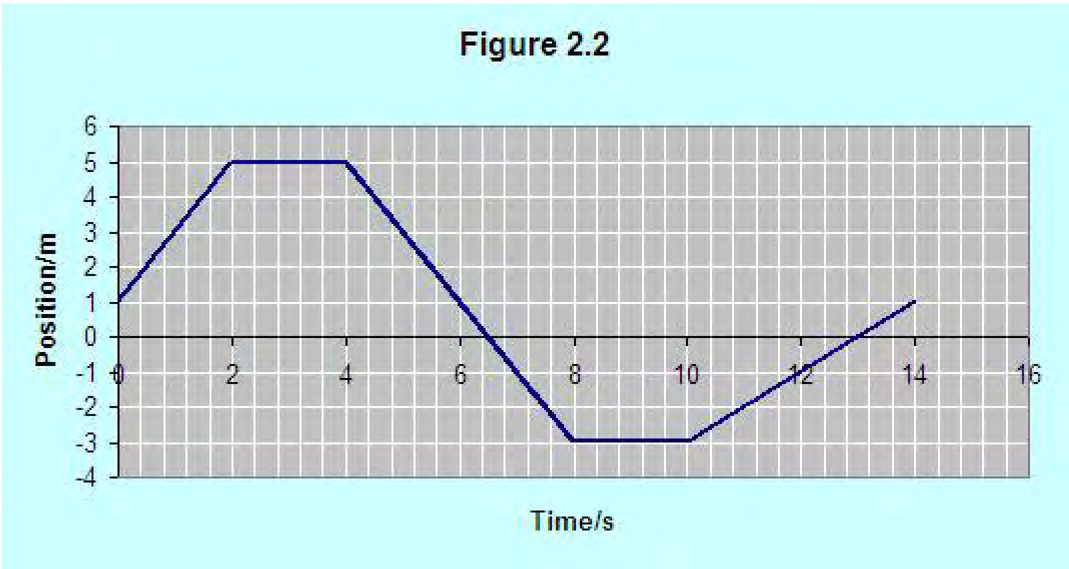

Figure 2.2

a) What is the initial position of the particle?

Solution: Initial position is position at $t = 0$.

$t = 0$; $x = ?$

$$x = 1 \text{ m}$$

b) Calculate the average velocity between $t = 2$ s and $t = 10$ s.

Solution: The average velocity is equal to the slope of the line joining the points $(2$ s, 5 m) and $(10$ s, -3 m).

$(t_i , x_i) = (2$ s, 5 m); $(t_f , x_f) = (10$ s, -3 m); $v_{av} = ?$

$$v_{av} = (x_f - x_i)/(t_f - t_i) = (-3 - 5)/(10 - 2) \text{ m/s} = -1 \text{ m/s}$$

c) Calculate the average velocity for the entire trip.

The average velocity for the entire trip is the slope of the line joining the points $(0, 1$ m$)$ and $(14$ s, 1 m).

$(t_i , x_i) = (0, 1$ m); $(t_f , x_f) = (14$ s, 1 m); $v_{av} = ?$

$$v_{av} = (x_f - x_i)/(t_f - t_i) = (1 - 1)/(14 - 0) \text{ m/s} = 0$$

d) Calculate the displacement of the particle between $t = 3$ s and $t = 9$ s.
Solution: $x_i = 5$ m; $x_f = -3$ m; $\Delta x = ?$

$$\Delta x = x_f - x_i = (-3 - 5) \text{ m} = -8 \text{ m}$$

e) On what interval(s) of time is the particle

i. at rest.

Solution: When the particle is at rest, the graph of position versus time should be horizontal. Therefore the particle is at rest on the time intervals between $t = 2$ s and $t = 4$ s and between $t = 8$ s and $t = 10$ s.

ii. moving to the right?

Solution: When moving to the right its velocity (slope) should be positive. Therefore the particle is moving to the right on the time intervals between $t = 0$ and $t = 2$ s and between $t = 10$ s and $t = 14$ s.

iii. moving to the left?

Solution: When moving to the left its velocity (slope) should be negative. Therefore the particle is moving to the left on the time interval between $t = 4$ s and $t = 8$ s

---

f) What is the instantaneous velocity of the particle at $t = 1$ s

Solution: If the graph of position versus time is a straight line, average and instantaneous velocity are the same. Since the graph is a straight line between $t = 0$ and $t = 2$ s, the instantaneous velocity at $t = 1$ s is equal to the average velocity between $t = 0$ and $t = 2$ s.

Solution: $(t_i , x_i) = (0,\ 1$ m); $(t_f , x_f) = (2$ s , 5 m); $v = v_{av} = ?$

$$v = (x_f - x_i)/(t_f - t_i) = (5 - 1)/(2 - 0)\ \text{m/s} = 2\ \text{m/s}$$

2.1.5 Average Acceleration

Average acceleration (a_{av}) is defined to be change in velocity per a unit time.

$$a_{av} = (v_f - v_i)/\Delta t$$

Where v_f is final velocity and v_i is initial velocity. The SI unit of measurement for acceleration is meter per second per second (m/s^2). Average acceleration between two events may be obtained from a graph of velocity versus time as the slope of the line joining the two events. Displacement also can be obtained from the graph of velocity versus time as the area enclosed between the velocity versus time curve and the time axis. Areas above the time axis are taken to be positive while areas below the time axis are taken to be negative.

Example: The velocity of a particle changed from 5 cm/s left to 7 cm/s right in 4 seconds. Calculate its average acceleration.

Solution: $v_i = -5$ cm/s; $v_f = 7$ cm/s; $\Delta t = 4$ s.

$$a_{av} = (v_f - v_{i)}/\Delta t = (7 - (-5))/4 \text{ cm/s}^2 = 3 \text{ cm/s}.$$

2.1.6 Instantaneous Acceleration

Instantaneous acceleration (a) is defined to be acceleration at a given instant of time. In other words, it is average acceleration evaluated at a very small interval of time. Instantaneous acceleration at a given instant of time may be obtained from a graph of velocity versus time as the slope of the line tangent to the curve at the given point.

Example: The following is a graph of velocity versus time for a certain particle.

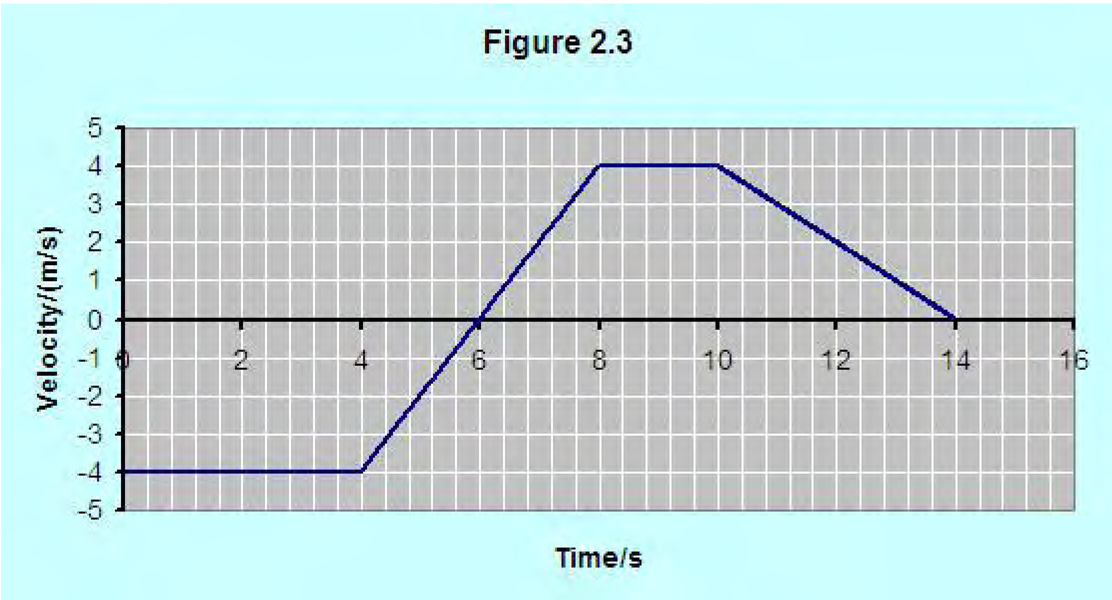

Figure 2.3

a) Calculate the average acceleration for the first ten seconds.

Solution: The average acceleration in the first ten seconds is equal to the slope of the line joining the events *(0, -4* m/s) and *(10* s, *4* m/s).

Solution: $(t_i, v_f) = (0, -4$ m/s); $(t_f, v_f) = (10$ s, 4 m/s); $a_{av} = ?$

$$a_{av} = (v_f - v_i)/(t_f - t_i) = (4 - (-4))/(10 - 0) \text{ m/s}^2 = 0.8 \text{ m/s}^2$$

---

b) Calculate the average acceleration for the entire trip.

Solution: The average acceleration for the entire trip is equal to the slope of the line joining the points *(0, -4* m) and *(14* s, *0)*.

Solution: $(t_i, v_f) = (0, -4$ m/s); $(t_f, v_f) = (14$ s, $0)$; $a_{av} = ?$

$$a_{av} = (v_f - v_i)/(t_f - t_i) = (0 - (-4))/(14 - 0) \text{ m/s}^2 = 0.286 \text{ m/s}^2$$

---

c) On what interval(s) of time is the particle

i. moving to the right?

Solution: The particle is moving to the right when the velocity is positive. Therefore the particle is moving to the right on the interval between $t = 6$ s and $t = 14$ s.

---

ii. moving to the left?

Solution: The particle is moving to the left when the velocity is negative. Therefore the particle is moving to the left on the interval between $t = 0$ and $t = 6$ s.

---

iii. temporarily at rest?

Solution: The particle is temporarily at rest when the velocity is zero. Therefore the particle is temporarily at rest at $t = 6$ s and $t = 14$ s.

---

d) On what interval(s) of time is the particle

i. moving with a constant velocity?

Solution: The particle moves with a constant velocity when its acceleration is zero or when the graph of velocity versus time is a horizontal line. Therefore the particle is moving with a constant velocity on the time intervals between $t = 4$ s and $t = 8$ s and between $t = 8$ s and $t = 10$ s.

ii. increasing its velocity?

Solution: The velocity of the particle increases when the acceleration or slope is positive. Therefore its velocity is increasing on the interval between $t = 4$ s and $t = 8$ s.

iii. decelerating?

Solution: The particle decelerates (its velocity decreases) when its acceleration or slope is negative. Therefore it is decelerating on the time interval between $t = 10$ s and $t = 14$ s.

---

e) Calculate its instantaneous acceleration at $t = 2$ s.

Solution: At $t = 2$ s, the tangent line is horizontal and the slope of a horizontal line is zero. Therefore the instantaneous acceleration at $t = 2$ s is zero.

2.2 Practice Quiz 2.1

Choose the best answer. Answers can be found at the back of the book.

1. A physical quantity that represents change in position is called
   A. distance
   B. displacement
   C. position
   D. speed
   E. velocity

2. The SI unit of measurement for position is
   A. meter
   B. meter * second
   C. meter / second
   D. meter / second ^{2}
   E. meter ^{2}

3. Instantaneous acceleration at a given event can be obtained from a (1 point)
   A. position versus time as the slope of the line tangent to the curve at the given point
   B. velocity versus position as the slope of the line tangent to the curve at the given point
   C. acceleration versus time as the slope of the line tangent to the curve at the given point
   D. velocity versus time as the slope of the line tangent to the curve at the given point
   E. acceleration versus position as the slope of the line tangent to the curve at the given point

4. This question is based on Figure 2.1. Each unit is one centimeter If the reference point is fixed at point F, obtain the positions of points A, F, and H respectively.
   A. *0* cm, -*5* cm, and -*7* cm
   B. -*5* cm, *0* cm, and *2* cm
   C. *0* cm, *5* cm, and *7* cm
   D. *5* cm, *0* cm, and -*2* cm
   E. *5* cm, *0* cm, and *2* cm

5. This question is based on Figure 2.1. Each unit is *1* cm. Particle 1 is displaced from point B to point F and particle 2 is displaced from point D to point A. The displacements of particles 1 and 2 respectively are
   A. -*4* cm, -*3* cm
   B. -*4* cm, *3* cm
   C. *5* cm, -*4* cm
   D. *4* cm, *3* cm
   E. *4* cm, -*3* cm

6. Calculate the average velocity of a particle displaced from the point $x = 8$ cm to the point $x = 9$ cm in *6* seconds.
   A. *0.183* cm / s
   B. *0.167* cm / s
   C. *0.217* cm / s
   D. *0.233* cm / s
   E. *0.2* cm / s

7. A particle is displaced from the point $x = 1$ cm to the point $x = 19$ cm. And then it is displaced back to the point *10* cm. Determine its displacement and the distance it travelled respectively.
   A. *-9* cm, *27* cm
   B. *9* cm, *27* cm
   C. *10* cm, *29* cm
   D. *11* cm, *18* cm
   E. *10* cm, *47* cm

8. A particle is displaced from the point $x = 7$ cm to the point $x = 17$ cm in *5* seconds. And then it is displaced back to the point *10* cm in *20* seconds. Calculate its average velocity and its average speed respectively.
   A. *0.22* cm / s, *1.02* cm / s
   B. *0.2* cm / s, *1.133* cm / s
   C. *0.12* cm / s, *0.68* cm / s
   D. *0.105* cm / s, *0.68* cm / s
   E. *0.12* cm / s, *0.826* cm / s

9. This question is based on Figure 2.2. Calculate the average velocity for the entire trip.
   A. *0* m / s
   B. *0.571* m / s
   C. *14* m / s
   D. *-0.571* m / s
   E. *1* m / s

10. This question is based on Figure 2.2. On what interval(s) of time is the particle moving to the left?
   A. *0-2* s, and *10-14* s
   B. *4-8* s
   C. *0-2* s, *4-8* s, and *10-14* s
   D. *10-14* s
   E. *0-8* s

11. This question is based on Figure 2.2. Calculate the instantaneous velocity after *6* seconds.
    - A. *-1* m/s
    - B. *2* m/s
    - C. *-2* m/s
    - D. *0* m/s
    - E. *1* m/s

12. This question is based on Figure 2.3. Calculate its average acceleration for the first 8 seconds.
    - A. *0.5* m/s^2
    - B. *-0.5* m/s^2
    - C. *1* m/s^2
    - D. *2* m/s^2
    - E. *-1.5* m/s^2

13. This question is based on Figure 2.3 Calculate its displacement between the 6^{th} and 8^{th} seconds.
    - A. *6* m
    - B. *5* m
    - C. *4* m
    - D. *8* m
    - E. *3* m

2.3 Uniformly Accelerated Motion

Uniformly accelerated motion is motion with a constant acceleration. It is motion where the rate of change of velocity with time is constant. Since the acceleration is a constant, average and instantaneous acceleration are the same. Thus, since $a_{av} = (v_f - v_i)/t$ (Assuming $t_i = 0$ we may write $\Delta t = t$), $a = (v_f - v_i)/t$. Or

$$v_f = v_i + at$$

Since the velocity changes uniformly, the average velocity of a uniformly accelerated motion is equal to the average of its initial and final velocities.

$$v_{av} = (v_i + v_f)/2$$

And since $\Delta x = v_{av}\, t$,

$$\Delta x = (v_i + v_f)(t/2)$$

Two other equations can be obtained by substituting for v_f and for t into this equation. Replacing v_f by $(v_i + at)$, the following equation can be obtained.

$$\Delta x = v_i\, t + (a/2)t^2$$

And replacing t by $(v_f - v_i)/a$, the following equation can be obtained.

$$v_f^{\,2} = v_i^2 + 2a\Delta x$$

The following is a list of the four equations of a uniformly accelerated motion:

$$v_f = v_i + at$$

$$\Delta x = v_i t + (a/2)t^2$$

$$v_f^{\,2} = v_i^2 + 2a\Delta x$$

$$\Delta x = (v_i + v_f)(t/2)$$

Only two of these equations are independent. These equations involve five variables: t, a, Δx, v_i, and v_f. Since only two of these equations are independent, if any three of these variables are known, we can usually solve for the other two by using suitable equations.

Example: The speed of a car changed from *20* m / s to *40* m / s in 10 seconds.

a) Calculate its acceleration.

Solution: $t = 10$ s; $v_i = 20$ m / s; $v_f = 40$ m / s; $a = ?$

$$v_f = v_i + at$$

$$a = (v_f - v_i)/t = (40 - 20)/10 \text{ m/s}^2 = 2 \text{ m/s}^2$$

---

b) Calculate the distance travelled.

Solution:

$$\Delta x = v_i\, t + (a/2)t^2 = \{(20)(10) + (2/2)(10^2)\} \text{ m} = 300 \text{ m.}$$

Example: A car initially moving with a speed of *50* m / s was stopped in a distance of *100* m.

a) Calculate its acceleration.

Solution: $v_i = 50$ m / s; $v_f = 0$; $\Delta x = 100$ m; $a = ?$

$$v_f^{\,2} = v_i^2 + 2a\Delta x$$

$$a = (v_f^{\,2} - v_i^2)/(2\Delta x) = (0^2 - 50^2)/2/\Delta x) \text{ m/s}^2 = -12.5 \text{ m/s}^2$$

---

b) Calculate the time taken.

Solution:

$$v_f = v_i + at$$

$$t = (v_f - v_i)/a = (0 - 50)/(-12.5) \text{ s} = 4 \text{ s.}$$

2.3.1 Motion under Gravity

Motion under gravity is a uniformly accelerated motion. The absolute value of gravitational acceleration (g) is *9.8* m / s². Since the effect of gravitational acceleration is to decrease velocity (decrease a positive velocity when going up and increase a negative velocity when coming down), gravitational acceleration is negative.

$$g = -9.8 \text{ m/s}^2$$

2.3.2 Equations of Motion under Gravity

Since gravitational motion is a uniformly accelerated motion, its equations can be obtained from the equations of a uniformly accelerated motion by replacing a by g. Also, since gravitational motion is a vertical motion, Δx needs to be replaced by Δy. Δy is taken to be positive if the final position is above the initial position and negative if the final position is below the initial position. The following are equations of gravitational motion.

$$v_f = v_i + gt$$

$$\Delta y = v_i t + (g/2)t^2$$

$$v_f^{\,2} = v_i^2 + 2g\Delta y$$

$$\Delta y = (v_i + v_f)(t/2)$$

These equations involve four variables. If any two of these variables are known, we can usually solve for the other variables.

Example: A ball is dropped from a height of *20* m.

a) Calculate the time taken to reach the ground.

Solution: $v_i = 0$ (because it is dropped from rest); $\Delta y = -20$ m (negative, because it is going down); $t = ?$

$$\Delta y = v_i\, t + (g/2)t^2 = (g/2)t^2 \text{ (since } v_i = 0 \text{ m/s)}$$

$$t = \sqrt{(2\Delta y/g)} = \sqrt{\{2(-20)/(-9.8)\}} \text{ s} \approx 2 \text{ s}$$

---

b) Calculate its speed by the time it hits the ground.

Solution: $v_f = ?$

$$v_f = v_i + at = \{0 + (-9.8)(2)\} \text{ m/s} \approx = -20 \text{ m/s}$$

Example: A ball is thrown upward with a speed of *20* m/s.

a) Calculate the time taken to reach the maximum height.

Solution: $v_i = 20$ m/s; $v_f = 0$ m/s (At maximum height speed is zero).

$$v_f = v_i + at$$

$$t = (v_f - v_i)/g = (0 - 20)/(-9.8) \approx 2 \text{ s}$$

---

b) To what height would it rise?

Solution: $\Delta y = ?$

$$\Delta y = (v_i + v_f)(t/2) = (20 + 0)(2/2) \text{ m} \approx 20 \text{ m}$$

Example: A ball is thrown upwards from a 10 m tall building with a speed of 10 m / s.

a) Calculate the speed with which it will hit the ground.

Solution: $v_i = 10$ m / s; $\Delta y = -10$ m (negative because the final position is below the initial position); $v_f = ?$

$$v_f^2 = v_i^2 + 2g\Delta y$$

$$v_f = \sqrt{v_i^2 + 2g\Delta y} = -\sqrt{10^2 + 2(-9.8)(-10)} \text{ m/s} \approx -17.3 \text{ m/s}$$

The negative square root is taken because it is going down.

---

b) Calculate the time taken to hit the ground.

Solution: $t = ?$

$$v_f = v_i + gt$$

$$t = (v_f - v_i)/g = (-17.3 - 10)/(-9.8) \text{ s} \approx 2.73 \text{ s}$$

2.4 Practice Quiz 2.2

Choose the best answer. Answers can be found at the back of the book.

1. The value of gravitational acceleration in the vicinity of the surface of earth is
   A. 9.8 m / s^2
   B. -9.8 m / s^2
   C. 98 cm / s^2
   D. -98 cm / s^2
   E. -32 m / s^2

2. A particle, starting from a certain speed, was accelerated for 10 seconds to a speed of 18 m / s with an acceleration of 0.6 m / s^2. Calculate its initial speed.
   A. 179.4 m / s
   B. 24 m / s
   C. 29.616 m / s
   D. 14.808 m / s
   E. 12 m / s

3. A particle, starting from a speed of 1.7 m/s, was accelerated with a uniform acceleration for 4 seconds. If it was displaced by 49 m during this time interval, calculate its acceleration.
   A. 5.913 m/s^2
   B. 21.1 m/s^2
   C. 1.319 m/s^2
   D. 5.275 m/s^2
   E. 6.975 m/s^2

4. A particle, initially moving with a speed of 34 m/s, was accelerated through a distance of 58 m with a uniform acceleration of 81 m/s^2. Calculate its speed at the end of the trip.
   A. 36.304 m/s
   B. 97.108 m/s
   C. 102.723 m/s
   D. 68.79 m/s
   E. 76.511 m/s

5. A particle initially moving with a speed of *8* m / s was accelerated uniformly for *0.8* seconds. Calculate its speed after it has travelled a distance of *80* m.
   A. *192* m / s
   B. *256.672* m / s
   C. *236.928* m / s
   D. *392* m / s
   E. *108* m / s

6. The speed of a particle changed from *21* m / s to *59* m / s in *30* seconds with a uniform acceleration. Calculate the acceleration and the distance travelled respectively.
   A. *1.267* m / s^2, *570* m
   B. *2.667* m / s^2, *1200* m
   C. *2.667* m / s^2, *570* m
   D. *2.533* m / s^2, *2400* m
   E. *1.267* m / s^2, *1200* m

7. A particle, starting from a speed of *10* m / s, accelerated for *15* seconds with a uniform acceleration of *4* m / s^2. The distance travelled and its final speed respectively are
   A. *1050* m and *70* m / s
   B. *1050* m and *910* m / s
   C. *600* m and *70* m / s
   D. *180* m and *40* m / s
   E. *600* m and *910* m / s

8. The speed of a uniformly accelerated particle changed from *800* m / s to *100* m / s in a distance of *250* m. Its acceleration and the time taken respectively are
   A. *-1.4* m / s^2 and *0.278* s
   B. *-1260* m / s^2 and *0.556* s
   C. *-1.4* m / s^2 and *0.556* s
   D. *1300* m / s^2 and *0.278* s
   E. *-1260* m / s^2 and *0.278* s

9. An object released from a certain height took *2* seconds to heat the ground. The height and its velocity just before it hits respectively are
   A. *9.8* m and *-19.6* m / s
   B. *39.2* m and *-39.2* m / s
   C. *19.6* m and *-9.8* m / s
   D. *19.6* m and *-19.6* m / s
   E. *9.8* m and *-9.8* m / s

10. An object is thrown upwards with a speed of *17* m / s. The maximum height reached and the time taken to reach the maximum height respectively are
    A. *29.49* m and *0.867* s
    B. *0.867* m and *3.469* s
    C. *14.745* m and *1.735* s
    D. *0.867* m and *1.735* s
    E. *14.745* m and *3.469* s

11. An object is thrown upwards from a *28* m building with a speed of *35* m / s. Its velocity just before it hits the ground and the time taken to hit the ground respectively are
    A. *-42.117* m / s and *6.037* s
    B. *-38.722* m / s and *7.338* s
    C. *-24.162* m / s and *6.037* s
    D. *-42.117* m / s and *7.869* s
    E. *-24.162* m / s and *7.869* s

3 Two Dimensional Motion

Your goals for this chapter are to learn about the variables used to describe motion in a plane and how they are related to each other.

Two dimensional motion is motion in a plane. Specification of the motion variables requires a pair of numbers. The kind of algebra used to operate variables represented by a pair of numbers is called vector algebra.

3.1 Vectors

Physical quantities are classified into vectors and scalars based on how they are represented. Scalars are physical quantities that can be specified by a number and a unit completely. For example when we say the length of an object *5* m, the number *5* and the unit m specify the length of the object completely. Therefore, length is a scalar quantity. Other examples of scalar physical quantities are mass, time, temperature and volume. Vectors are physical quantities that require the specification of direction in addition to a number and a unit. For example velocity is a vector quantity because, in addition to how fast an object is going, we also need to know in what direction it is going. Other examples of vectors are displacement, acceleration, force and area.

3.1.1 Graphical Representation of Vectors

In text books, a vector is symbolically represented by a letter in bold face while in exercise books, it is represented by a capital letter with an arrow on top. Graphically, a vector is represented by means of an arrow. The direction of the vector is represented by the direction of the arrow. The magnitude (numerical value) of the vector is represented by the length of the arrow. The arrow is drawn in such a way that its length is proportional to the magnitude of the vector. This is done by specifying a scale and drawing the length according to the scale.

The negative of a vector is the vector with the same magnitude but opposite direction. For example if **A** = *4* m east, then -**A** = *4* m west. A vector can be multiplied by a number. The effect of multiplying a vector by a number is to multiply the magnitude of the vector. If the number is positive the direction remains the same. If the number is negative the direction becomes opposite. For example if **A** = *2* m north, then *4***A** = *8* m north and -*4***A** = *8* m south.

3.1.2 Adding Vectors Graphically

The sum of two or more vectors (sometimes called the resultant) is the single vector with the same effect. For example, suppose a particle is displaced *4* m east from point A to point B. And then again displaced *4* m north from point B to point C. This can be accomplished by a single vector displacing the particle from point A to point C directly in a north east direction. The latter is the sum of the two vectors.

To add vectors graphically, first connect all the vectors tail to head. Then the sum vector is the vector whose tail is the tail of the first vector and whose head is the head of the last vector. The magnitude of the sum vector is obtained by measuring the length and multiplying by the scale. The direction is obtained by measuring the angle between the sum vector and the positive x-axis by means of a protractor.

3.1.3 Subtracting Vectors Graphically

Subtracting vector **B** from vector **A** means adding the negative of **B** to **A**. That is **A** – **B** = **A** + (-**B**). To subtract vector **B** from **A** graphically, first find the negative of vector **B**. Then add vector **A** and the negative of vector **B** using the rules of addition.

3.1.4 Algebraic Representation of Vectors

Algebraically a vector may be represented in terms of polar coordinates or Cartesian coordinates.

Polar Coordinates

The polar coordinates of a vector are its magnitude and its direction which is specified as the angle between the positive x-axis (east or horizontal line to the right of the tail of the vector) and the vector which is measured in a counterclockwise direction from the positive x-axis. For example the angles for east, north, west and south are *0°, 90°, 180°* and *270°* respectively. Angles measured in a clockwise direction are taken to be negative. For example *270°* and *-90°* represent the same angle which is south.

Example: Determine the magnitude and the direction of the following vectors.

a) *300* m/s west

Solution: Its magnitude is *300* m/s and its direction as measured from positive x-axis (east) is *180°*.

---

b) *400* m *30°* west of north

Solution: The magnitude is *400* m. Since the angle as measured from north to the west is *30°* and the angle for north with respect to east is *90°*, the angle for this vector must be *120°*

---

c) *60* m *40°* south of east.

Solution: The magnitude is *60* m. This vector makes an angle of *320°* with the positive x-axis when measured in a counterclockwise direction or *40°* when measured in a clockwise direction. Thus, this angle can be represented either as *320°* or *-40°*.

---

d) *7* m/s² at *200°*

Solution: The magnitude is *7* m/s². The default reference line is east. So, if no reference line is specified, the reference line is east. Thus, the angle for this vector is *200°*

Cartesian Coordinates

The Cartesian coordinates of a vector are its projection on the x-axis (horizontal line) and its projection on the y-axis (vertical line). The projection on the x-axis is called the x-component or the horizontal component of the vector. The projection on the y-axis is called the y-component or the vertical component of the vector. Components can be positive or negative. A horizontal component is taken to be positive if it is to the right and negative f it is to the left. A vertical component is taken to be positive if its direction is up and negative if its direction is down. For example, for a displacement vector, *5* m *37°* north of east, the projections on the x-axis and y-axis (as can be shown by dropping the perpendiculars on the x-axis and the y-axis) are 4m and *3* m respectively.

3.1.5 Obtaining Horizontal and Vertical Components from Magnitude and Direction

The magnitude of a vector will be represented by the symbol of the vector in italics. For example, the magnitude of the vector **A** will be represented by *A*. The direction of a vector will be represented by θ. The x-component (y-component) of a vector will be represented by the symbol of the vector with subscript x (y) in italics. For example, the x- and y- components of the vector **A** will be represented by A_x and A_y respectively.

For simplicity, consider a vector **A** on the first quadrant (whose angle is less than *90* °). The vector and its horizontal and vertical components form a right angled triangle. The hypotenuse of this triangle is equal to the magnitude of the vector, *A*. The horizontal component, A_x, is adjacent to the angle of the vector, θ. The vertical component, A_y, is opposite to the angle of the vector, θ.

The definitions of cosine and sine can be used to obtain the following expressions for the horizontal and vertical components of the vector in terms of the magnitude and direction of the vector:

$$A_x = A \cos \theta$$

$$A_y = A \sin \theta$$

Example: Find the horizontal and vertical components of the following vectors.

a) **A** = *100* m *53°*; north of east.

Solution: $A = 100$ m; $\theta = 53°$ $A_x = ?$; $A_y = ?$.

$$A_x = A \cos \theta = 100 \text{ m} \cos 53° = 60 \text{ m}$$

$$A_y = A \sin \theta = 100 \text{ m} \sin 53° = 80 \text{ m}$$

---

b) **A** = *10* m *30°* south of west.

Solution: $A = 10$ m; $\theta = 180° + 30° = 210°$; $A_x = ?$; $A_y = ?$

$$A_x = A \cos \theta = 10 \text{ m} \cos 210° = -5 \text{ m}$$

$$A_y = A \sin \theta = 10 \text{ m} \sin 210° \approx -8.7 \text{ m.}$$

3.1.6 Obtaining Magnitude and Direction of a Vector from the Components of the Vector

Applying Pythagorean Theorem to the right angled triangle considered earlier, the following equation for the magnitude can be obtained in terms of the components of the vector.

$$A = \sqrt{(A_x^2 + A_y^2)}$$

The angle θ can be related to the component with the help of the trigonometric function tangent: $\tan\theta = A_y/A_x$. And an expression for θ can be obtained by applying arctan to both sides of this equation.

$$\theta = \arctan(A_y/A_x)$$

The calculator will give only values between $-90°$ and $90°$ because the period of tangent is $180°$ and not $360°$. And thus, if A_x is negative $180°$ should be added to the angle obtained from the calculator. If A_x is zero, the calculator will give an error because division by zero is not allowed. If A_x is zero and A_y is positive, the value of θ is $90°$. If A_x is zero and A_y is negative the value of θ is $-90°$ or $270°$.

Example: Calculate the magnitude and direction of the vector whose horizontal and vertical components (respectively) are.

a) 3 m and 4 m

Solution: $A_x = 3$ m; $A_y = 4$ m; $A = ?$; $\theta = ?$

$$A = \sqrt{(A_x^2 + A_y^2)} = \sqrt{(3^2 + 4^2)}\ \text{m} = 5\ \text{m}.$$

$$\theta = \arctan(A_y/A_x) = \arctan(4/3) = 53°$$

---

b) -3 m and 4 m

Solution: $A_x = -3$ m; $A_y = 4$ m; $A = ?$; $\theta = ?$.

$$A = \sqrt{(A_x^2 + A_y^2)} = \sqrt{\{(-3)^2 + (-4)^2\}}\ \text{m} = 5\ \text{m}$$

$$\theta = \arctan(A_y/A_x) + 180° = \arctan\{(4)/(-3)\} + 180° = -53° + 180° = 127°$$

$180°$ is added because A_x is negative

3.1.7 Adding Vectors Algebraically

The horizontal component of the sum of two or more vectors is equal to the sum of the horizontal components of the vectors. Also, the vertical component of the sum vector is equal to the sum of the vertical components of the vectors.

If

$$\mathbf{R} = \mathbf{A} + \mathbf{B}$$

Then

$$R_x = A_x + B_x$$

$$R_y = A_y + B_y$$

The magnitude and direction of the sum vector can be obtained from these components.

$$R = \sqrt{(R_x^2 + R_y^2)} = \sqrt{\{(A_x + B_x)^2 + (A_y + B_y)^2\}}$$

$$\theta = \arctan\,(R_y/R_x) = \arctan\,\{(A_y + B_y)/(A_x + B_x)\}$$

Example: Given the vectors **A** = *100* m *37°* north of east and **B** = *100* m *53°* south of west.

a) Calculate the horizontal and vertical components of their sum vector

Solution: $A = 100$ m; $\theta_A = 37°$; $B = 100$ m; $\theta_B = 53° + 180° = 233°$ (180° is added to find the angle with respect to the positive x-axis); $R_x = A_x + B_x = ?$; $R_y = A_y + B_y = ?$

$$A_x = A \cos \theta_A = 100 \text{ m} \cos 37° = 80 \text{ m}$$

$$A_y = A \sin \theta_A = 100 \text{ m} \sin 37° = 60 \text{ m}$$

$$B_x = B \cos \theta_B = 100 \text{ m} \cos 233° = -60 \text{ m}$$

$$B_y = B \sin \theta_B = 100 \text{ m} \sin 233° = -80 \text{ m}$$

$$R_x = A_x + B_x = (80 - 60) \text{ m} = 20 \text{ m}$$

$$R_y = A_y + B_y = (60 - 80) \text{ m} = -20 \text{ m}$$

---

b. Calculate the magnitude and the direction of the sum vector.

Solution: $R_x = 20$ m; $R_y = -20$ m; $R = ?$; $\theta = ?$

$$R = \sqrt{(R_x^2 + R_y^2)} = \sqrt{\{20^2 + (-20)^2\}} \text{ m} \approx 28 \text{ m}$$

$$\theta = \arctan (R_y/R_x) = \arctan (-20/20) = -45°$$

3.2 Practice Quiz 3.1

Choose the best answer. Answers can be found at the back of the book.

1. Which of the following physical quantities is a vector?
   A. Velocity
   B. Temperature
   C. Speed
   D. Time
   E. Length

2. The default angle (angle measured with respect to the positive x-axis) for the vector $\boldsymbol{A}$ = *2* m north is
   A. *90* deg
   B. *-90* deg
   C. *180* deg
   D. *270* deg
   E. *0* deg

3. The default angle (angle measured with respect to the positive x-axis) for the vector $\boldsymbol{A}$ = *2* m *40* deg north of east is
   A. *140*
   B. *40* deg
   C. *-40* deg
   D. *130* deg
   E. *50* deg

4. Using the scale *1* cm for *10* m / s, the vector $\boldsymbol{A}$ = *24* m / s east, can be graphically represented by an arrow of length
   A. *24* cm
   B. *0.417* cm
   C. *4.8* cm
   D. *2.4* cm
   E. *1.2* cm

5. If the vector $\boldsymbol{A}$ = *80* m / s *55* deg west of south, using the scale *1* cm for *30* m / s, the vector *0.8*$\boldsymbol{A}$ can be graphically represented by an arrow of length _________ and default angle (angle measured with respect to the positive x-axis) ___________.
   A. *1.067* cm, *215* deg
   B. *2.133* cm, *235* deg
   C. *1.067* cm, *235* deg
   D. *2.133* cm, *215* deg
   E. *4.267* cm, *55* deg

6. If the vector $\boldsymbol{A}$ = *60* m / s south, using the scale *1* cm for *30* m / s, the vector *-0.8*$\boldsymbol{A}$ can be graphically represented by an arrow of length _________ and default angle (angle measured with respect to the positive x-axis) ___________.
   A. *1.6* cm, *-90* deg
   B. *0.8* cm, *90* deg
   C. *0.8* cm, *-90* deg
   D. *1.6* cm, *90* deg
   E. *0.5* cm, *90* deg

7. Given the vectors ***A*** = *10* m east and ***B*** = *3* m south, using graphical addition determine which of the following most likely represent the magnitude and direction of the sum vector (***A*** + ***B***) respectively. (You should be able to answer it with sketches not drawn to scale.)
   A. *10.44* m, *343.301* deg
   B. *13* m, *343.301* deg
   C. *10.44* m, *16.699* deg
   D. *10.44* m, *196.699* deg
   E. *13* m, *196.699* deg

8. Given the vectors ***A*** = *8* m west and ***B*** = *9* m north, using graphical method determine which of the following most likely represent the magnitude and direction of the difference vector, ***A*** – ***B***, respectively. (You should be able to answer it with sketches not drawn to scale.)
   A. *12.042* m, *48.366* deg
   B. *12.042* m, *228.366* deg
   C. *17* m, *311.634* deg
   D. *12.042* m, *311.634* deg
   E. *17* m, *228.366* deg

9. Determine the x and y components of the vector ***A*** = *100* m east.
   A. (*0* m, -*100* m)
   B. (*100* m, *100* m)
   C. (*0* m, *100* m)
   D. (*100* m, *0* m)
   E. (-*100* m, *0* m)

10. Determine the x and y components of the vector ***A*** = *440* m *30* deg east of south.
   A. (*264* m, -*381.051* m)
   B. (*264* m, -*304.841* m)
   C. (*220* m, -*381.051* m)
   D. (*220* m, -*304.841* m)
   E. (*252.374* m, *360.427* m)

11. Calculate the magnitude of a vector whose x and y components are *8* m and *5* m respectively.
   A. *13* m
   B. *3.606* m
   C. *89* m
   D. *9.434* m
   E. *8.307* m

12. Calculate the direction (angle with respect to the positive x axis) of a vector whose x and y components are *14* m and *13* m respectively.
    A. *38.591* deg
    B. *42.879* deg
    C. *47.167* deg
    D. *47.121* deg
    E. *51.833* deg

13. Calculate the magnitude and direction (with respect to the positive x axis) of a vector whose x and y components are *-20* m and *9* m respectively.
    A. *24.125* m, 155.772 deg
    B. *24.125* m, *150.772* deg
    C. *21.932* m, *-24.228* deg
    D. *21.932* m, *155.772* deg
    E. *21.932* m, *150.772* deg

14. Given the vectors ***A*** = *30* m *50* deg north of east and ***B*** = *80* m *60* deg south of west, find the x and y components of their sum vector ***A*** + ***B***.
    - A. (*-24.86* m, *-46.301* m)
    - B. (*-24.86* m, *73.811* m)
    - C. (*-29.003* m, *-27.78* m)
    - D. (*-20.716* m, *-46.301* m)
    - E. (*-20.716* m, *-37.041* m)

15. Given the vectors ***A*** = *70* m *10* deg north of east and ***B*** = *80* m *20* deg south of east, find the magnitude and direction (with respect to the positive x axis) of their sum vector ***A*** + ***B***.
    - A. *144.912* m, *-6.971* deg
    - B. *144.912* m, *-6.023* deg
    - C. *173.894* m, *-6.971* deg
    - D. *115.93* m, *-7.919* deg
    - E. *173.894* m, *-6.023* deg

3.3 Two Dimensional Motion Variables

Motion variables are position, displacement, velocity and acceleration.

3.3.1 Position

The *position vector* of a particle is defined to be the vector whose tail is at the origin of the coordinate plane and whose head is at the location of the particle. Position vector is customarily represented by **r**. If the Cartesian coordinate of the particle is (x, y), then the x-component of the position vector is equal to x and the y-component of the position vector is equal to y.

$$r_x = x$$

$$r_y = y$$

Example: A particle is located at a point whose ordered pair is $(-8, 6)$ m.

Calculate the x and y components of its position vector.

Solution: $(x, y) = (-8, 6)$ m; $x = -8$ m; $y = 6$ m; $r_x = ?$; $r_y = ?$

$$r_x = x = -8 \text{ m}$$

$$r_y = y = 6 \text{ m}$$

---

a) Calculate the magnitude and the direction of its position vector.

Solution: x = -8 m; $y = 6$ m; $r = ?$; $\theta = ?$

$$r = \sqrt{(x^2 + y^2)} = \sqrt{(-8)^2 + 6^2}\ \text{m} = 10\ \text{m}$$

$$\theta = \arctan\ (y/x) = \arctan\ (6/-8) = 143°$$

3.3.2 Displacement

Displacement is defined to be change in the position vector of a particle. Graphically, a displacement vector is a vector whose tail is the initial location of the particle and whose head is at the final location of the particle. Displacement is customarily represented by $\Delta\mathbf{r}$. If $\mathbf{r}_i$ is the initial position vector and $\mathbf{r}_f$ is the final position vector, then the displacement vector is given by:

$$\Delta\mathbf{r} = \mathbf{r}_f - \mathbf{r}_i$$

The horizontal (vertical) component of a displacement is equal to the difference between the horizontal (vertical) components of the final and initial position vector.

$$\Delta r_x = \Delta x = x_f - x_i$$

$$\Delta r_y = \Delta y = y_f - y_i$$

Example: A particle is displaced from the point $(4, -2)$ m to the point $(8, -6)$ m.

a) Calculate the horizontal and vertical components of its displacement vectors.

Solution: $x_i = 4$ m; $y_i = -2$ m; $x_f = 8$ m; $y_f = -6$ m; $\Delta r_x = ?$; $\Delta r_y = ?$

$$\Delta r_x = x_f - x_i = (8 - 4)\ \text{m} = 4\ \text{m}$$

$$\Delta r_y = y_f - y_i = (-6 - (-2))\ \text{m} = -4\ \text{m}$$

b) Calculate the magnitude and direction of the displacement vector.

Solution:

$$\Delta r = \sqrt{(\Delta x^2 + \Delta y^2)} = \sqrt{4^2 + (-4)^2}\ \text{m} \approx 5.66$$

$$\theta = \arctan\ (\Delta y/\Delta x) = \arctan\ (-4/4) = -45°$$

3.3.3 Average Velocity

Average velocity is defined to be displacement per a unit time.

$$\mathbf{v}_{av} = \Delta\mathbf{r}/\Delta t$$

The horizontal (vertical) component of average velocity is equal to the horizontal (vertical) component of displacement vector per a unit time.

$$v_{avx} = \Delta x/\Delta t$$

$$v_{avy} = \Delta y/\Delta t$$

Example: A particle is displaced from the point $(3, 6)$ m to the point $(-5, 6)$ m in 2 seconds. Calculate the horizontal and vertical components of its average velocity.

Solution: $x_i = 3$ m; $y_i = 6$ m; $x_f = -5$ m; $y_f = 6$ m; $\Delta t = 2$ s; $v_{avx} = ?$; $v_{avy} = ?$

$$v_{avx} = (x_f - x_i)/\Delta t = (-5 - 3)/2 \text{ m/s} = -4 \text{ m/s}.$$

$$v_{avy} = (y_f - y_i)/\Delta t = (6 - 6)/2 \text{ m/s} = 0 \text{ m/s}.$$

3.3.4 Instantaneous Velocity

Instantaneous velocity is velocity at a given instant of time. Or, it is average velocity evaluated at a very small interval of time.

Average Acceleration

Average Acceleration is defined to be change in velocity per a unit time.

$$\mathbf{a}_{av} = \Delta\boldsymbol{v}/\Delta t = (\mathbf{v}_f - \mathbf{v}_i)/\Delta t$$

Instantaneous Acceleration

Instantaneous acceleration is acceleration at a given instant of time. It is average acceleration evaluated at a very small interval of time.

3.4 Uniformly Accelerated Motion

Uniformly accelerated motion is motion with constant acceleration. The horizontal and vertical components of the acceleration are constant. The horizontal and vertical components of the motion are independent of each other in a sense that what goes on horizontally does not affect what goes on vertically or vice versa. Therefore, a two dimensional motion can be decomposed into a one dimensional horizontal motion and a one dimensional vertical motion. The one dimensional equations of a uniformly accelerated motion obtained earlier are applicable to these one dimensional components of a two dimensional motion. A very popular example of a uniformly accelerated two dimensional motion is gravitational motion in a curved path commonly referred as projectile motion.

3.4.1 Projectile Motion

Motion under gravity is a uniformly accelerated motion. The magnitude of gravitational acceleration is 9.8 m/s² and is directed downwards. Thus, the components of gravitational acceleration are:

$$a_x = 0 \text{ m/s}^2$$

$$a_y = g = -9.8 \text{ m/s}^2$$

The horizontal component of the motion is a non-accelerated motion. The equations that govern this component of the motion are equations of uniform motion. The horizontal component of the velocity remains the same since there is no acceleration in this direction. The horizontal component of the displacement is obtained as the product of the horizontal component of the velocity and time taken. (That is, equation of a uniform motion)

$$v_{fx} = v_{ix}$$

$$\Delta x = v_{ix} t$$

The vertical component of the motion is a uniformly accelerated motion with an acceleration g. The equations governing this component of this motion are the already obtained equations of a one dimensional uniformly accelerated motion.

$$v_{fy} = v_{iy} + gt$$

$$\Delta y = v_{iy}t + (g/2)t^2$$

$$v_{fy}^2 = v_{iy}^2 + 2g\Delta y$$

$$\Delta y = (v_{iy} + v_{fy})(t/2)$$

Before using these equations, an initial point and a final point should be chosen. t is the time taken for the object to go from the initial point to the final point. Time is always positive. Δx and Δy are the horizontal and vertical components of the displacement vector whose tail is at the initial point and whose head is at the final point. Appropriate signs should be used. Δx is taken to be positive if the final point is to the right of the initial point and negative if the final point is to the left of the initial point. Δy is taken to be positive if the final point is above the initial point and negative if the final point is below the initial point. v_{ix} and v_{iy} are the horizontal and vertical components of the velocity at the initial point. v_{fx} and v_{fy} are horizontal and vertical components of the velocity at the final point. A horizontal component of a velocity is positive if it points to the right and negative if it points to the left. A vertical component of a velocity is positive if it points up and negative if it points down.

Example: A ball is thrown from a *2* m table horizontally (to the right) with a velocity of *10* m/s.

a) Calculate the horizontal and vertical components of the initial velocity.

Solution: $v_i = 10$ m/s; $\Delta y = -2$ m; $\theta_i = 0°$; $v_{ix} = ?$; $v_{iy} = ?$

$$v_{ix} = v_i \cos\theta_i = 10 \text{ m/s} \cos 0° = 10 \text{ m/s}$$

$$v_{iy} = v_i \sin\theta_i = 10 \text{ m/s} \sin 0° = 0 \text{ m/s}$$

---

b) Calculate the time taken to hit the ground.

Solution: $t = ?$

$$\Delta y = v_{iy}t + (g/2)t^2 = (g/2)t^2$$

Because $v_{iy} = 0$ m/s

$$t = \sqrt{(2\Delta y/g)} = \sqrt{\{(2)(-2)/(-9.8)\}} \approx 0.64 \text{ m/s}$$

---

c) Calculate the horizontal and vertical components of its velocity by the time it hits the ground.

Solution: $v_{fx} = ?$; $v_{fy} = ?$

$$v_{fx} = v_{ix} = 10 \text{ m/s}$$

$$v_{fy} = v_{iy} + gt = \{0 + (-9.8)(0.0.64)\} \text{ m/s} = -6.26 \text{ m/s}$$

---

d) How far will it fall (horizontally)?

Solution: Δx = ?

$$\Delta x = v_{ix}t = (10)(0.64) \text{ m/s} = 6.4 \text{ m}$$

Example: A bullet is fired from the ground with a speed of *1000* m/s making an angle of *37*° with the horizontal.

a) Calculate the horizontal and vertical components of its initial velocity.

Solution: $v_i = 1000$ m/s; $\theta_i = 37°$; $v_{ix} = ?$; $v_{iy} = ?$

$$v_{ix} = v_i \cos \theta_i = (1000 \cos 37°) \text{ m/s} = 800 \text{ m/s}$$

$$v_{iy} = v_i \sin \theta_i = (1000 \sin 37°) \text{ m/s} = 600 \text{ m/s}$$

---

b) Calculate the time taken to reach the maximum height.

Solution: At the maximum height the direction of the velocity at the maximum height is horizontal which means the vertical component of the velocity at the maximum height is zero: $v_{fy} = 0$ m/s; $t = ?$

$$v_{fy} = v_{iy} + gt = 0 \text{ m/s}$$

$$t = (-v_{iy})/g = (-600)/(-9.8) \text{ s} = 61.22 \text{ s}$$

---

c) Calculate the maximum height reached.

Solution: $\Delta y = ?$

$$\Delta y = v_{iy}t + (g/2)t^2 = ((600)(61.22) + (-9.8/2)61.22^2) \text{ m} = 18367.35 \text{ m}$$

---

d) How far will it fall (horizontally)?

Solution: The time taken to return to the ground is twice the time taken to reach the maximum height: $t = 2(61.22)$ s $= 122.44$ s; $\Delta x = ?$

$$\Delta x = v_{ix}t = (800*122.44) \text{ m} = 97952 \text{ m}$$

3.5 Practice Quiz 3.2

Choose the best answer. Answers can be found at the back of the book.

1. Which of the following is not a correct statement?
   A. The x component of the displacement vector of a particle is equal to the change in the x coordinate of the particle.
   B. The magnitude of the displacement vector of a particle is equal to the length of the hypotenuse of a right angled triangle whose base and height are equal to the change in its x and y coordinates respectively.
   C. Displacement and distance are not the same.
   D. The displacement vector of a particle is the vector whose tail is at the final location of the particle and whose head is at the initial location of the particle.
   E. The magnitude of a displacement vector is equal to the distance between its initial location and final location.

2. A particle is displaced from the point (*-2* m, *-40* m) to the point (*-5* m, *10* m). Calculate the vertical component of its displacement vector.
   A. *-50* m
   B. *3* m
   C. *50* m
   D. *-3* m
   E. *-30* m

3. A particle is displaced from the point (4 m, *14* m) to the point (6 m, *7* m) in *25* seconds. Calculate the direction (with respect to the positive x axis) of its average velocity vector.
   A. *-81.46* deg
   B. *-59.244* deg
   C. *-74.055* deg
   D. *-96.271* deg
   E. *-88.866* deg

4. A bullet is fired horizontally from a *30* m tall building with a speed of *800* m / s. How long will it take to hit the ground?
   A. *2.474* s
   B. *1.485* s
   C. *1.979* s
   D. *1.732* s
   E. *2.227* s

5. A bullet is fired horizontally from a *40* m tall building with a speed of *1500* m / s. Calculate the vertical component of its velocity just before it hits the ground?
   A. -*30.8* m / s
   B. -*36.4* m / s
   C. -*28* m / s
   D. -*39.2* m / s
   E. -*33.6* m / s

6. A bullet is fired horizontally from a *80* m tall building with a speed of *1000* m / s. How far (horizontally) will it fall?
   A. *3232.488* m
   B. *4040.61* m
   C. *4444.671* m
   D. *3636.549* m
   E. *4848.732* m

7. A bullet is fired from the ground making an angle of *20* deg with the horizontal with a speed of *800* m / s. How long will it take to reach its maximum height?
   A. *22.336* s
   B. *30.712* s
   C. *27.92* s
   D. *25.128* s
   E. *19.544* s

8. A bullet is fired from the ground making an angle of *60* deg with the horizontal with a speed of *1400* m / s. Calculate the maximum height reached?
   A. *82500* m
   B. *97500* m
   C. *67500* m
   D. *75000* m
   E. *90000* m

9. A bullet is fired from the ground making an angle of *10* deg with the horizontal with a speed of *1000* m / s. How far (horizontally) will it fall?
   A. *31410.013* m
   B. *27920.012* m
   C. *41880.018* m
   D. *34900.015* m
   E. *38390.016* m

10. A bullet is fired from a *80* m tall building with a speed of *1300* m / s making an angle of *10* deg with the horizontal. Calculate the vertical component of its velocity by the time it hits the ground.
    A. *-275.027* m / s
    B. *-252.108* m / s
    C. *-297.946* m / s
    D. *-229.189* m / s
    E. *-206.27* m / s

4 Laws of Motion

Your goals for this chapter are to learn about the nature of forces and the laws relating forces with motion.

Laws of motion are laws that establish relationship between force and motion. Force is a physical quantity that changes the motion or the shape of an object. Or in layman terms, force is a push or a pull. The SI unit of force is the Newton abbreviated as N. There are three laws that establish relationship between force and motion which were discovered by the British scientist Isaac Newton.

Newton's First Law states that an object stays at rest or continues to move in a straight line with a constant speed unless acted upon by a net force. This law describes what happens to an object when there is no force acting on it. A net force is required to change the direction of motion or to accelerate an object. No force is required to keep an object going in a straight line with a constant speed. There must be a net force acting on an object moving in a circular path with a constant speed because its direction is changing. There must be a net force acting on an object falling in a straight line because its speed is changing. It is a common experience that when a driver applies the brakes suddenly, his (her) body tends to move forward. This is because the force exerted by his (her) back sit ceases to act after the brakes are applied and his (her) body tends to move in a straight line with the same speed he (she) had when the brakes were just applied. Of course other forces will stop this motion shortly.

Newton's second Law states that the net force acting on an object is directly proportional to its acceleration. The constant of proportionality between the force acting on an object and its acceleration is a measure of the amount of matter the object has and is called the mass of the object.

$$F_{net} = ma$$

F_{net} is the net force acting on an object of mass m moving with acceleration a. The SI unit of mass is the kilo gram (kg). The unit of force (N) is equal to the product of the unit mass (kg) and the unit of acceleration (m/s^2).

Example: Calculate the acceleration of an object of mass *20* kg when acted upon by a force of *100* N.

Solution: $F_{net} = 100$ N; $m = 20$ kg; $a = ?$

$$F_{net} = ma$$

$$a = F_{net}/m = (100/20) \text{ m/s}^2 = 5 \text{ m/s}^2$$

Example : A force that causes an acceleration of 3 m/s² when acting on object A of mass 5 kg is acting on object B of mass 30 kg. Calculate the acceleration of object B.

Solution: $m_A = 5$ kg; $a_A = 3$ m/s²; $m_B = 30$ kg; $F_{net} = ?$

$$F_{net} = m_A a_A = 5*3 \text{ N} = 15 \text{ N}$$

$$F_{net} = m_B a_B$$

$$a_B = F_{net}/m_B = 15/30 \text{ m/s}^2 = 0.5 \text{ m/s}^2$$

Newton's third law states that for any action there is an equal but opposite reaction. If object A exerts a force on object B, then object B exerts a force on object A which is of the same magnitude but opposite in direction.

$$\mathbf{F}_{AB} = -\mathbf{F}_{BA}$$

$\mathbf{F}_{BA}$ is force exerted on object B by object A and $\mathbf{F}_{AB}$ is force exerted by on object A by object B. Action reaction forces act on different objects. An object cannot exert force on itself directly. An object can exert force on itself only by exerting force on another object so that the reaction force acts on it. If one wants to walk forward, he (she) has to push on the ground backward so that the reaction force of the ground pushes him (her) forward. If one wants to swim forward, he (she) has to push on the water backwards so that the reaction force of the water pushes him (her) forward. If a car is to move forward, the wheels of the car must push on the ground backwards so that the reaction force of the ground pushes the car forward. If a rocket is to be propelled into space, it has to push on a gas downward so that the reaction force of the gas propels it upward.

Example: Object A exerts a force of 8 N east on object B. Determine the magnitude and the direction of the force exerted by object B on object A.

Solution: $\mathbf{F}_{BA} = 8$ N east; $\mathbf{F}_{AB} = ?$

$$\mathbf{F}_{AB} = -\mathbf{F}_{BA} = 8 \text{ N west}$$

4.1 Types of Forces

Forces may be generally classified as contact and non-contact forces.

4.2 Non-contact Forces

Non-contact forces are forces that objects exert on each other without a direct contact between the objects. For example earth can attract an object in its vicinity towards itself even though there is no direct contact between earth and the object. This kind of force is called gravitational force. Other non-contact forces include electrical and magnetic forces. In this course, we will deal only with gravitational force.

4.2.1 Weight

Weight of an object in the vicinity of a massive object is the gravitational force exerted by the massive object on the object. For example, the weight of an object in the vicinity of earth is the gravitational force exerted on the object by earth. The acceleration of a falling object, which is the acceleration due to the gravitational force exerted by earth, is a constant and is equal to $|g| = 9.8$ m/s². Therefore the weight of an object on the vicinity of earth is equal to the product of its mass and this gravitational acceleration.

$$w = m|g|$$

Where w is the weight of an object of mass m. In the vicinity of earth, weight and mass are proportional; which makes it possible to measure mass by measuring weight. But mass and weight are very different physical properties. Mass is a measure of the amount of matter the object has. It is an inherent property of the object. Its value does not depend on other objects in its vicinity. An object will have the same mass on all planets. On the other hand, weight is a measure of the gravitational force due to massive objects in its vicinity. Its value depends on the mass of the massive object in its vicinity. An object will have different weights on different planets. For example the weight of an object on the moon is one sixth of its weight on earth, because gravitational acceleration on the moon is one sixth of that on earth. The unit of weight is the unit of force which is the Newton.

Example: Calculate the weight of a *100* kg object.

Solution: m = 100 kg; w = ?

$$w = m|g| = 100*9.8 \text{ N} = 980 \text{ N}$$

Example: An object of weight *100* N is being acted upon by a net force of *20*. Calculate its acceleration.

Solution: w = 100 N; $F_{net} = 20$ N; a = ?

$$w = m|g|$$

$$m = w/|g| = 100/9.8 \text{ kg} = 10.2 \text{ kg}$$

$$F_{net} = ma$$

$$a = F_{net}/m = 20/10.2 \text{ m/s}^2 = 1.96 \text{ m/s}^2$$

4.3 Contact Forces

Contact forces are forces exerted between objects in direct contact with each other. These kind of forces can be identified easily by identifying the objects in contact with the object under consideration. But there is one kind of contact force that deserves a special attention. This is force between two surfaces in contact.

4.3.1 Surface Forces

Surface forces are forces between surfaces in contact. A surface force is customarily decomposed into a component that is parallel to the surfaces and a component that is perpendicular to the surfaces. The component that is parallel to the surfaces is called the force of friction. When the two surfaces slide on each other, this force opposes the motion. The direction of this force on one of the surfaces is opposite to the direction of the sliding motion with respect to the other surface. The component that is perpendicular to both surfaces is the force that presses the two surfaces together and is called the normal force. Without this force, the two surfaces will not be in contact. It is a common experience that the greater the force that presses the two surfaces together (the normal force), the more difficult it is to slide one of the surfaces relative to the other surface. In other words, the greater the normal force, the greater the force of friction. Experiment shows that the force of friction and the normal force are directly proportional. The constant of proportionality is called the coefficient of friction between the two surfaces. It is unit less.

$$f = \mu N$$

Where f is the force of friction, N is the normal force and μ is the coefficient of friction between the surfaces. There are two kinds of coefficients of friction. Experiment shows that the amount of force required to just get an object sliding is greater than the force required to keep it sliding once it has started sliding. This means two types of coefficient of friction are required. The coefficient of friction related with the force of friction just before it starts sliding is called the static coefficient of friction.

$$f = \mu_s N$$

Where μ_s is the static coefficient of friction between the surfaces. The coefficient of friction related with the force of friction when the two surfaces are sliding with respect to each other is called the coefficient of kinetic friction.

$$f = \mu_k N$$

Where μ_k is the kinetic coefficient of friction between the surfaces. Since the force of friction just before an object starts sliding is greater than the force after it starts sliding, coefficient of static friction is greater than coefficient of kinetic friction.

$$\mu_s > \mu_k$$

Coefficient of friction depends on the types of surfaces in contact. For example sliding on a rough surface is much harder than sliding on a smooth surface. Coefficient of friction does not depend on the area of the surface area of contact. A rectangular solid object may have surfaces of different surface areas. All of the surfaces will have the same coefficient of friction. Also, coefficient of friction does not depend on the relative speed between the surfaces. Whether the relative speed between the surfaces is *10* m/s or *20* m/s coefficient of friction will always be the same.

Example: An object of mass *10* kg is sliding in a horizontal surface with a uniform speed. The coefficient of kinetic friction is *0.2*.

a) Calculate the normal force pressing the two surfaces together.

   Solution: $m = 10$ kg; $\mu_k = 0.2$; $N = ?$

   The net force acting on the object in the direction perpendicular to the surfaces must be zero, because the object is not moving in this direction. There are two forces acting on this object in this direction. One is the weight of the object which is directed downward. The other force is the vertical component of the force exerted by the surface on which it is sliding. This is the normal force. Since the net force in this direction is zero, these two forces must be balancing each other.

$$N = w = m|g| = 10*9.8 \text{ N} = 98 \text{ N}$$

---

b) Calculate the force of friction.

   Solution: $f = ?$

$$f = \mu_k N = 0.2*98 \text{ N} = 1.96 \text{ N}$$

Example: An object of mass 10 kg is being pulled on a horizontal surface with a horizontal force of 100 N. The coefficient of kinetic friction is 0.3.

a) Calculate the normal force pressing the two surfaces together.

Solution: $m = 10$ kg; $F = 100$ N (F is the horizontal force pulling to the right); $\mu_k = 0.3$; $N = ?$

The net vertical force acting on the object must be zero because the object is not moving in this direction. Therefore the vertical forces acting on this object (normal force and its weight) must balance each other.

$$N = w = m|g| = 10 * 9.8 \text{ N} = 98$$

---

b) Calculate the force of friction resisting the motion of the object.

Solution: $f = ?$

$$f = \mu_k N = 0.3 * 98 = 29.4 \text{ N}$$

---

c) Calculate the acceleration of the object.

Solution: a = ?

Horizontally, there are two forces acting on the object. One is the horizontal force pulling on the object (F). The other is the force of friction (f) opposing this force. The net horizontal force is the difference between these forces.

$$F_{net} = F - f = (100 - 29.4)\text{ N} = 70.6\text{ N}$$

$$F_{net} = ma$$

$$a = F_{net}/m = 70.6/10\text{ m/s}^2 = 7.06\text{ m/s}^2$$

Example: Consider a book on the top of a table. Identify four action reaction pairs.

Solution:

a) Gravitational force exerted by earth on book and gravitational force exerted by book on earth.
b) Gravitational force exerted by earth on table and gravitational force exerted by table on earth.
c) Surface force exerted by table on book and surface force exerted by book on table.
d) Surface force exerted by ground on table legs and surface force by table legs on ground

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4.4 Practice Quiz 4.1

Choose the best answer

1. State Newton's second law.
   A. The force acting on an object is inversely proportional to the acceleration produced by the force.
   B. Any two objects in the universe attract each other with a force directly proportional to the product of their masses and inversely proportional to the square of the distance separating them.
   C. For every reaction, there is an equal but opposite reaction.
   D. The force acting on an object is directly proportional to the acceleration produced by the force.
   E. An object will remain at rest or move in a straight line with a constant speed unless acted upon by a net force.

2. Which of the following situations can be explained by Newton's second law?
   A. When the force acting on an object is doubled, its acceleration is doubled.
   B. When a man fires a gun, his body jerks backwards.
   C. A man walking forward while pushing on the ground backwards.
   D. When a driver applies the breaks suddenly, his body jerks forward.
   E. A feather falling in a straight line with a constant speed.

3. Which of the following statements is correct.
   A. Weight and mass have the same unit of measurement.
   B. Mass has different values on different planets; weight has the same value on different planets.
   C. Weight has different values on different planets; mass has the same value on different planets.
   D. Mass has the same value on different planets; weight has the same value on different planets.
   E. Mass has different values at different planets; weight has different values at different planets.

4. An object has a mass of 50 kg. Calculate its acceleration when acted upon by a force of 200 N.
   A. $3.2\ \mathrm{m/s^2}$
   B. $4\ \mathrm{m/s^2}$
   C. $2.4\ \mathrm{m/s^2}$
   D. $4.4\ \mathrm{m/s^2}$
   E. $4.8\ \mathrm{m/s^2}$

5. Calculate the mass of an object whose weight is 400 N.
   A. 44.898 kg
   B. 40.816 kg
   C. 32.653 kg
   D. 28.571 kg
   E. 48.98 kg

6. Under the influence of a certain force the velocity of a 10 kg object increased from $16\ \mathrm{m/s}$ to $50\ \mathrm{m/s}$ in 80 seconds. Calculate the force exerted on the object.
   A. 4.25 N
   B. 3.4 N
   C. 2.55 N
   D. 2.975 N
   E. 4.675 N

7. The component of the surface force between two surfaces sliding on each other perpendicular to the surface is called
   A. contact force
   B. weight
   C. normal force
   D. friction
   E. gravitational force

8. Which of the following is a correct statement
   A. Direction of force of friction is always parallel to the direction of motion.
   B. Force of friction does not depend on the normal force.
   C. Force of friction does not depend on the relative speed between the surfaces sliding on each other.
   D. Force of friction does not depend on the nature of the surfaces in contact.
   E. Force of friction depends on the area of the surfaces in contact.

9. An object of mass *88* kg is sliding on a horizontal surface. Calculate the normal force exerted by the surface on the object.
   A. *862.4* N
   B. *1034.88* N
   C. *948.64* N
   D. *689.92* N
   E. *776.16* N

10. An object of mass *30* kg is sliding on a horizontal surface with a uniform speed. The coefficient of kinetic fiction of the surfaces is *0.15*. Calculate the force of friction exerted by the surface on the object.
   A. *294* N
   B. *61.74* N
   C. *57.33* N
   D. *44.1* N
   E. *26.46* N

11. An object of mass *9* kg is being pulled by a horizontal force of *280* N on a horizontal force. The coefficient of kinetic friction between the object and the surface is *0.35*. Calculate the acceleration of the object.
   A. *24.913* m/s^2
   B. *22.145* m/s^2
   C. *27.681* m/s^2
   D. *33.217* m/s^2
   E. *30.449* m/s^2

4.5 Statics

Statics is the study of objects in equilibrium. An object is said to be in equilibrium (translational) if it is either at rest or moving in a straight line with a constant speed. According to Newton's first law, this happens when the net force acting on the object is zero.

4.5.1 Condition of Equilibrium

The *condition of equilibrium* states that an object will be in equilibrium (translational) if the sum of all the forces acting on the object is zero.

$$\mathbf{F}_{net} = \Sigma\,\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \ldots = 0$$

Where $\mathbf{F}_1$, $\mathbf{F}_2$, … are all the forces acting on the object. This is a vector equation. It can be decomposed into an equation for the horizontal components and an equation for the vertical components to give two algebraic equations. If a vector is equal to zero, then its components are also equal to zero.

$$\Sigma F_x = F_{1x} + F_{2x} + \dots = 0$$

$$\Sigma F_y = F_{1y} + F_{2y} + \dots = 0$$

Where $F_{1x} + F_{2x} + \dots$ and $F_{1y} + F_{2y} + \dots$ are the horizontal and vertical components of all the forces acting on an object in equilibrium respectively.

4.5.2 Solving Equilibrium Problems

Identify all the forces acting on the object. Identifying a force most of the time means identifying the magnitude and the direction of the force. In identifying direction, it is a good practice to obtain angles measured from the default reference line (east or positive x-axis). Representing the force vectors graphically helps to visualize the problem better. The contact forces can easily be identified by identifying the objects in contact with the object under consideration. There is always the non-contact force weight. The direction of weight is downwards or towards the ground (its default angle is $-90°$). Its horizontal component is zero. Its vertical component is negative because it points downward.

After all the forces have been identified (along with the unknowns), Find the horizontal and vertical components of each force with the appropriate signs. If the default angle (angle with respect to the positive x-axis) is used, the formulas for calculating components will yield the appropriate sign. If angles with respect to other reference lines are used, the appropriate sign should be included according to the sign conventions for components. Magnitude is always positive. Plug in the components into the two equations of condition of equilibrium in the form of components and solve for the unknowns. These equations can be used to solve for two unknowns.

Example: A 10 kg object is hanging from a ceiling by means of a string. Calculate the tension in the string.

Solution: There are two forces acting on this object in equilibrium: its weight and the tension in the string. The direction of the force due to the tension in the string is upwards. Since both vectors are vertical only the equation for vertical components need to be used. Tension in the string will be represented by T.

$m = 10$ kg; $T = ?$; $\theta_T = 90°$; $\theta_w = -90°$; $T = ?$

$$w = m|g| = 10*9.8 \text{ N} = 98 \text{ N}$$

$$w_y = w \cos \theta_w = 98*\sin(-90°) = -98$$

$$T_y = T \sin \theta = T \sin (90°) = T$$

$$T_y + w_y = 0$$

$$T = T_y = -w_y = -(-98) \text{ N} = 98 \text{ N}$$

Example: A *100* kg object is hanging from a string by means of two strings. One of the strings is directed at *37°* north of east. The other string is directed at *53°* north of west. Calculate the tensions in the strings.

Solution: There are three forces acting on this object in equilibrium: its weight and the tensions in the two strings.

$m = 100$ kg; $\theta_{T1} = 37°$; $\theta_{T2} = (180 - 53)° = 127°$; $T_1 = ?$; $T_2 = ?$

$$w = m|g| = 100{*}9.8 \text{ N} = 980 \text{ N}$$

$$w_x = w \cos \theta_w = 980 \cos -90° \text{ N} = 0 \text{ N}$$

$$w_y = w \sin \theta_w = 980 \sin -90° \text{ N} = -980 \text{ N}$$

$$T_{1x} = T_1 \cos \theta_{T1} = T_1 \cos 37° = 0.8T_1$$

$$T_{1y} = T_1 \sin \theta_{T1} = T_1 \sin 37° = 0.6T_1$$

$$T_{2x} = T_2 \cos \theta_{T2} = T_2 \cos 127° = -0.6T_2$$

$$T_{2y} = T_2 \sin \theta_{T2} = T_2 \sin 127° = 0.8T_1$$

$$T_{1x} + T_{2x} + w_x = 0.8T_1 - 0.6T_2 + 0 \text{ N} = 0 \text{ N}$$

$$T_1 = 0.75T_2$$

$$T_{1y} + T_{2y} + w_y = 0.6T_1 + 0.8T_2 + -980 \text{ N} = 0 \text{ N}$$

$$0.6(0.75T_2) + 0.8T_2 = 980 \text{ N}$$

$$T_2 = 980/(0.6*0.75 + 0.8) \text{ N} = 784 \text{ N}$$

$$T_1 = 0.75T_2 = 0.75*784 \text{ N} = 588 \text{ N}$$

4.6 Dynamics

Dynamics is the study of accelerated systems. According to Newton's second law, the product of the mass of an object and its acceleration is equal to the net force acting on the object.

$$\mathbf{F}_{net} = \Sigma\, \mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \dots = m\mathbf{a}$$

Where $\mathbf{F}_1 + \mathbf{F}_2 + \dots$ are all the forces acting on an object of mass m moving with an acceleration of $\mathbf{a}$. This is a vector equation. It can be decomposed into two one dimensional algebraic equations: one relating the horizontal components and the other relating the vertical components. This is possible because the horizontal and vertical motions are independent in a sense that acceleration in one doesn't affect motion in the other.

$$\Sigma\, F_x = F_{1x} + F_{2x} + \dots = ma_x$$

$$\Sigma\, F_y = F_{1y} + F_{2y} + \dots = ma_y$$

Where $F_{1x} + F_{2x} + \dots$ and $F_{1y} + F_{2y} + \dots$ are the horizontal and vertical components of all the forces acting on the object respectively.

4.6.1 Solving Dynamics problems

Identify all the forces acting on the object. The non-contact force, weight, is always there. The contact forces can easily be determined by looking at the objects in contact with the object. If the object is sliding on a surface, make sure to include the force exerted by the surface. The vertical component (normal) and the horizontal component (friction) of a surface force are customarily dealt with separately. The normal force is perpendicular to both surfaces and is directed towards the surface of the object under consideration. If the object is moving in a straight path, the direction of friction is parallel to the surfaces and opposes the direction of motion.

Find the horizontal and vertical components of all the forces (including unknown forces). In calculating components, it is a good idea to use the angle measured with respect to the positive x-axis so that you don't have to worry about having the right sign for the components. Magnitude is always positive. Weight is directed towards the ground. Its horizontal component is zero. Its vertical component is negative and is numerically equal to the weight. Its default angle is *-90°*. For an object sliding on a horizontal surface, the default angle for the normal force is *90°*. Its horizontal component is zero. Its vertical component is positive and is equal to the magnitude of the normal force. Again, if the object is sliding in a straight path, the direction of friction is parallel to the surfaces and opposes the direction of motion. Its default angle is 180° (assuming the motion is along the positive x-axis). Its vertical component is zero. Its horizontal component is negative and is numerically equal to the magnitude of friction. Most of the time, you will be dealing with horizontal motion in a straight line. In such cases, the vertical component of acceleration is zero and the horizontal component is numerically equal to the magnitude of acceleration.

Example: An object of mass *20* kg is moving on a frictionless horizontal surface. It is being pulled by a string that makes an angle of 60° with the horizontal. The tension in the string is *100* N. Calculate the acceleration of the object.

Solution: The forces acting on the object are its weight, the tension in the string and the force exerted by the surface in which it is sliding. Since the surface is frictionless, the surface force consists of the normal force only. The vertical component of the acceleration is zero because it is moving on a horizontal surface. The only force with a horizontal component is the tension in the string which is already known. Thus, only the horizontal component of Newton's law needs to be considered.

$m = 20$ kg; $T = 100$ N; $\theta_T = 60°$; $a = ?$

$$a_x = a$$

$$T_x = T \cos \theta_T = (100 \cos 60°) \text{ N} = 50 \text{ N}$$

$$T_x = ma_x = ma$$

$$a = T_x / m = 50/20 \text{ m/s}^2 = 2.5 \text{ m/s}^2$$

Example: A *10* kg object is moving in a frictionless horizontal surface. It is being pulled by a *20* N force that makes angle of *37°* with the horizontal. It is also being pulled backwards by a horizontal force of *5* N. Calculate the acceleration of the object.

Solution: The forces acting on this object are its weight, the normal force exerted by the surface (friction is zero because the surface is frictionless) and the tensions in the two strings. The acceleration is horizontal because it is moving in a horizontal surface. The only forces with horizontal components are the tensions in the strings. Hence, only the horizontal component of Newton's second law needs to be used.

$m = 10$ kg; $T_1 = 20$ N; $\theta_{T1} = 37°$; $T_2 = 5$ N; $\theta_{T2} = 180°$

$$a = a_x$$

$$T_{1x} = T_1 \cos\theta_{T1} = (20 \cos 37°) \text{ N} = 16 \text{ N}$$

$$T_{2x} = T_2 \cos\theta_{T2} = (5 \cos 180°) \text{ N} = -5 \text{ N}$$

$$T_{1x} + T_{2x} = ma_x = ma$$

$$a = (T_{1x} + T_{2x})/m = \{(16 + (-5))/10\}\ \mathrm{m/s^2} = 1.1\ \mathrm{m/s^2}$$

Example: A *10* kg object is being pulled by a string that makes an angle of *30°* with the horizontal. The tension in the string is *100* N. The coefficient of kinetic friction between the surfaces is *0.2*.

a) Calculate the normal force exerted by the surface of the ground on the object.

Solution: The forces acting on the object are its weight, surface force which includes both friction and normal force and the tension in the string. The normal force is a vertical force directed upward. It has only a vertical component which is equal to the magnitude of the normal force. The other forces with vertical components are its weight and the tension in the string which are already known. Thus, only the vertical component of Newton's second law can be used to calculate the normal force.

$m = 10$ kg; $T = 100$ N; $\theta_T = 30°$; $N = ?$

$$N_y = N$$

$$w_y = -w = -m|g| = -10*9.8\ \mathrm{N} = -98\ \mathrm{N}$$

$$T_y = T \sin \theta_T = (100 \sin 30°)\ \mathrm{N} = 50\ \mathrm{N}$$

$$N_y + w_y + T_y = 0\ \mathrm{N}$$

$$N = N_y = -(w_y + T_y) = -(-98 + 50)\ \mathrm{N} = 48\ \mathrm{N}$$

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b. Calculate the force of friction.

Solution: $\mu = 0.2$; $f = ?$

$$f = \mu N = 0.2*48\ \mathrm{N} = 9.6\ \mathrm{N}$$

---

c. Calculate its acceleration.

Solution: The acceleration has only a horizontal component which is equal to the magnitude of acceleration (assuming it is going to the right) because it is moving horizontally. The forces with horizontal components are friction, and the tension in the string. The angle for friction is *180°* because it opposes the motion.

$\theta_f = 180°$; $a = ?$

$$a_x = a$$

$$f_x = f \cos \theta_f = (9.6 \cos 180°) \text{ N} = -9.6 \text{ N}$$

$$T_x = T \cos \theta_T = (100 \cos 30°) \text{ N} = 87 \text{ N}$$

$$T_x + f_x = ma_x = ma$$

$$a = (T_x + f_x) / m = \{(87 + (-9.6)) / 10 \} \text{ m/s}^2 = 7.74 \text{ m/s}^2$$

4.6.2 The Inclined Plane

Consider an object of mass m sliding in an inclined plane whose inclination with respect to the horizontal is θ. The force responsible for the sliding motion of the object is the component of gravitational force along the plane. Since gravitational force is vertical, the angle between the plane and the direction of its weight is $(90 - \theta)$. The sine of an angle is equal to the cosine of its complementary angle. The cosine of an angle is equal to the sine of its complementary angle. Thus, the component of gravitational force parallel to the plane is given by sine and the component perpendicular to the plane is given by cosine.

$$w_{||} = m|g| \sin \theta$$

$$w_{\perp} = m|g| \cos \theta$$

$w_{||}$ is the component of gravitational force parallel to the plane. $w_{\perp}$ is the component of gravity perpendicular to the plane. If the inclined plane is frictionless, the acceleration down the plane can be obtained by dividing the component of its weight along the plane by its mass.

$$a = (m|g| \sin \theta) / m = |g| \sin \theta$$

a is the acceleration of the object down the plane when friction can be neglected.

Example: A 10 kg object is sliding down a 30° inclined plane.

a) Calculate the component of gravitational force down the plane.

   Solution: $m = 10$ kg; $\theta = 30°$; $w_{||} = ?$

$$w_{||} = m|g| \sin \theta = (10 * 9.8 \sin 30°) \text{ N} = 49 \text{ N}$$

---

b) Calculate the component of gravitational force perpendicular to the plane.

Solution: $w_\perp = ?$

$$w_\perp = m|g|\cos\theta = (10*9.8\cos 30°)\ \text{N} = 85.3\ \text{N}$$

---

c) Calculate its acceleration down the plane.

Solution: $a_\parallel = ?$

$$a_\parallel = |g|\sin\theta = (9.8\sin 30°)\ \text{m/s}^2 = 4.9\ \text{m/s}^2$$

4.6.2 Systems Involving more than One Objects

Newton's second law can be applied provided only external forces are considered. External forces are forces exerted on the system by objects outside the system. A force exerted by one of the objects in the system on another object in the system is an internal force and should not be considered.

$$\Sigma F_{ext} = m_{total}a$$

Example: A *2* kg object is connected with an *8* kg object on a frictionless surface by means of a string. The *8* kg object is being pulled horizontally by a force of *100* N. Calculate the acceleration of the system.

Solution: $F_{ext} = 100$ N; $m_{total} = (2 + 8)$ kg $= 10$ kg; $a = ?$

$$F_{ext} = m_{total}a$$

$$a = F_{ext}/m_{total} = 100/10\ \text{m/s}^2 = 10\ \text{m/s}^2$$

4.7 Practice Quiz 4.2

Choose the best answer

1. An object is being pulled on a horizontal surface with a horizontal force of *120* N. If it is moving with a uniform speed and the coefficient of friction between the sliding surfaces is *0.3*, calculate the normal force exerted by the surface on the object.
   A. *120* N
   B. *36* N
   C. *400* N
   D. *480* N
   E. *0.004* N

2. An object of mass *200* kg is hanging from a ceiling by means of two strings. The first string (T_1) makes an angle of *20* degree with the horizontal-right. The second string (T_2) makes an angle of *60* degree with the horizontal-left. The equations relating the x-components and the y-components of the forces acting on the object respectively are
   A. $0.342 * T_1 + 0.866 * T_2 = 0$
      $-0.94 * T_1 + 0.5 * T_2 - 1960 \text{ N} = 0$
   B. $-0.94 * T_1 + 0.866 * T_2 = 0$
      $-0.94 * T_1 + 0.866 * T_2 - 1960 \text{ N} = 0$
   C. $-0.94 * T_1 + 0.5 * T_2 = 0$
      $0.342 * T_1 + 0.866 * T_2 - 1960 \text{ N} = 0$
   D. $0.342 * T_1 + 0.866 * T_2 - 1960 \text{ N} = 0$
      $-0.94 * T_1 + 0.5 * T_2 = 0$
   E. $-0.94 * T_1 + 0.5 * T_2 - 1960 \text{ N} = 0$
      $0.342 * T_1 + 0.866 * T_2 = 0$

3. An object of mass *600* kg is hanging from a ceiling by means of two strings. The first string (T_1) makes an angle of *25* degree with the horizontal-right. The second string (T_2) makes an angle of *35* degree with the horizontal-left. Calculate the tension in the first string (T_1)
   A. *5005.572* N
   B. *5561.747* N
   C. *4449.397* N
   D. *6674.096* N
   E. *6117.921* N

4. An object of mass *100* kg is being pulled on a friction less horizontal surface by means of a string that makes an angle of *30* degree with the horizontal. If the tension in the string is *60* N calculate the acceleration of the object.
   A. 0.6 m/s^2
   B. 0.52 m/s^2
   C. 9.8 m/s^2
   D. 0.346 m/s^2
   E. 0.3 m/s^2

5. An object of mass *32* kg is being pulled on a friction less horizontal surface by means of two strings. One of the strings has a tension of *54* N and makes an angle of *10* degree with the horizontal. The other string is pulling horizontally and has a tension of *2* N. Calculate the acceleration of the object.
   A. *2.242* $\mathrm{m/s^2}$
   B. *1.897* $\mathrm{m/s^2}$
   C. *2.414* $\mathrm{m/s^2}$
   D. *2.069* $\mathrm{m/s^2}$
   E. *1.724* $\mathrm{m/s^2}$

6. An object of mass *40* kg is being pulled on a friction less horizontal surface by means of two strings. One of the strings is pulling forward, has a tension of *240* N and makes an angle of *10* degree with the horizontal-right. The other string is pulling backwards horizontally and has a tension of *16* N. Calculate the acceleration of the object.
   A. *7.712* $\mathrm{m/s^2}$
   B. *4.407* $\mathrm{m/s^2}$
   C. *3.305* $\mathrm{m/s^2}$
   D. *6.611* $\mathrm{m/s^2}$
   E. *5.509* $\mathrm{m/s^2}$

7. An object of mass *6* kg is being pulled on a horizontal surface by a string that makes an angle of *50* degree with the horizontal. The tension in the string is *13* N. Calculate the normal force exerted by the ground on the object.
   A. *58.61* N
   B. *39.073* N
   C. *53.726* N
   D. *43.957* N
   E. *48.841* N

8. An object is of mass *48* kg is being pulled on a horizontal surface by a string that makes an angle of *10* deg with the horizontal. The tension in the string is *320* N. The coefficient of kinetic friction of the surfaces is *0.45*. Calculate the acceleration of the object.
   A. *2.409* $\mathrm{m/s^2}$
   B. *2.676* $\mathrm{m/s^2}$
   C. *2.141* $\mathrm{m/s^2}$
   D. *1.873* $\mathrm{m/s^2}$
   E. *2.944* $\mathrm{m/s^2}$

9. An object of mass *20* kg is sliding down a friction less inclined plane that makes an angle of *20* deg with the horizontal. Calculate the component of gravitational force that is pulling the object along the inclined plane.
   A. *196* N
   B. *184.18* N
   C. *3.352* N
   D. *9.209* N
   E. *67.036* N

10. A *4* kg object and a *7* kg object are attached by a string. If the *7* kg object is being pulled by a force of *200* N. calculate their acceleration.
    A. $14.545\ \mathrm{m/s^2}$
    B. $50\ \mathrm{m/s^2}$
    C. $28.571\ \mathrm{m/s^2}$
    D. $18.182\ \mathrm{m/s^2}$
    E. $1.855\ \mathrm{m/s^2}$

5 Work and Energy

Your goals for this chapter are to learn about work, energy, and the relationships between work and energy.

Work is a measure of a force's ability to displace or deform an object. The work done by a force in displacing an object is defined to be the product of the component of the force in the direction of the displacement and the magnitude of the displacement.

$$W = F_{||}\, d$$

Where W is the work done, $F_{||}$ is the component of the force in the direction of the displacement and d is the magnitude of the displacement. If the angle between the force and the displacement is θ, then the component of the force in the direction of the displacement is equal to the product of the magnitude of the force and the cosine of the angle θ.

$$W = Fd \cos \theta$$

Where F is the magnitude of the force. Work can be positive or negative depending on the value of θ. It is positive when θ is an acute angle and negative when θ is an obtuse angle. It is zero when the force is perpendicular to the displacement. The unit of measurement for work is the product of the unit of force (N) and unit of displacement (m) which is defined to be the Joule abbreviated as J.

Example: An object is pulled horizontally a distance of *5* m by a force of *100* N that makes an angle of 60° with the horizontal. Calculate the work done.

Solution: $d = 5$ m; $F = 100$ N; $\theta = 60°$; $W = ?$

$$W = Fd \cos \theta = (100 * 5 * \cos 60°)\text{ J} = 250\text{ J}$$

Example: An object is displaced horizontally a distance of *10* m while being pulled backwards by a force of *20* N that makes an angle of *37*° with the horizontal. (An object can move in a direction that opposes the force if it has an initial velocity. The effect of the force is to slow down the object). Calculate the work done.

Solution: $d = 10$ m; $F = 20$ N; $\theta = (180 - 37)° = 143°$; $W = ?$

$$W = Fd \cos \theta = (20 * 10 * \cos 143°)\text{ J} = -160\text{ J}$$

5.1 Work Kinetic Energy Theorem

The *net work done* on an object is the sum of all the works done by all the forces acting on the object. In other words, it is the work done by the net force acting on the object.

$$W_{net} = W_1 + W_2 + \ldots$$

Where W_{net} is the net work done on the object and W_1, W_2, … are the works done by the forces $\mathbf{F}_1$, $\mathbf{F}_2$, … respectively. Or

$$W_{net} = F_{net}\, d \cos \theta_{net}$$

Where F_{net} is the magnitude of the net force $\mathbf{F}_{net} = \mathbf{F}_1 + \mathbf{F}_2 + \ldots$ and θ_{net} is the angle between the displacement and the net force.

Example: A *10* kg object is displaced by a distance of *5* m on a horizontal frictionless surface under the influence of the following forces: A *10* N horizontal force, A *20* N force that makes an angle of *60°* with the horizontal and a *5* N force that is pulling backwards horizontally. Calculate the net work done on the object.

Solution: The forces acting on the object are its weight, normal force and the forces listed. The weight and the normal force are perpendicular to the displacement and hence the work done by its weight and the normal force does not contribute to the net work done.

$d = 5$ m; $F_1 = 10$ N; $\theta_1 = 0°$; $F_2 = 20$ N; $\theta_2 = 60°$; $F_3 = 5$ N; $\theta_3 = 180°$; $W_{net} = W_1 + W_2 + W_3 = ?$

$$W_1 = F_1\, d \cos \theta_1 = (10 * 5 * \cos 0°)\ \mathrm{J} = 50\ \mathrm{J}$$

$$W_2 = F_2\, d \cos \theta_2 = (20 * 5 * \cos 60)\ \mathrm{J} = 50\ \mathrm{J}$$

$$W_3 = F_3\, d \cos \theta_3 = (5 * 5 * \cos 180°)\ \mathrm{J} = -25\ \mathrm{J}$$

$$W_{net} = W_1 + W_2 + W_3 = (50 + 50 - 25)\ \mathrm{J} = 75\ \mathrm{J}$$

For simplicity let's assume that the net force acting on an object and its displacement are parallel.

$$W_{net} = F_{net}\, d$$

According to Newton's second Law $F_{net} = ma$.

$$W_{net} = mad$$

The product *ad* can be related to the initial and final speed of the object by using one of the equations of a uniformly accelerated motion.

$$v_f^2 = v_i^2 + 2ad$$

$$ad = (v_f^2 - v_i^2)/2$$

Substituting for *ad*, the following expression for the net work done is obtained.

$$W_{net} = mv_f^2/2 - mv_i^2/2$$

The expression $mv^2/2$ is defined to be the kinetic energy of an object of mass m moving with a speed v.

$$KE = mv^2/2$$

Where KE stands for kinetic energy. The unit of measurement for energy is the joule (J). With this definition of kinetic energy, the net work done on an object is equal to the change of its kinetic energy.

$$W_{net} = KE_f - KE_i = \Delta KE = mv_f^2/2 - mv_i^2/2$$

This is a mathematical statement of the work kinetic energy theorem. The *work kinetic energy theorem* states that the net work done on an object is equal to the change of its kinetic energy.

Example: Calculate the kinetic energy of an *8* kg moving with a speed of *5* m / s.

Solution: $m = 8$ kg; $v = 5$ m / s; $KE = ?$

$$KE = mv^2/2 = (8 * 5^2/2)\ \text{J} = 100\ \text{J}$$

Example: The speed of a *4* kg object changed from *10* m / s to *5* m / s. Calculate the net work done on the object.

Solution: $m = 4$ kg; $v_i = 10$ m / s; $v_f = 5$ m / s; $W_{net} = ?$

$$W_{net} = mv_f^2/2 - mv_i^2/2 = (4 * 5^2/2 - 4 * 10^2)\ \text{J} = -150\ \text{J}$$

Example: A *20* kg object is displaced by a distance of *2* m on a horizontal frictionless surface under the influence of the following forces: A *50* N horizontal force and a *20* N force that makes an angle of *37*° with the horizontal. If it started from rest,

a) calculate the net work done on the object.

   Solution: The forces acting on the object are its weight, the normal force exerted by the ground and the forces listed. Its weight and the normal force do not contribute to the net work done because they are perpendicular to the displacement.

   $d = 2$ m; $F_1 = 50$ N; $\theta_1 = 0°$; $F_2 = 20$ N; $\theta_2 = 37°$; $W_{net} = W_1 + W_2 = ?$

$$W_1 = F_1 d \cos\theta_1 = (50 * 2 * \cos 0°)\ \text{J} = 100\ \text{J}$$

$$W_2 = F_2 d \cos\theta_2 = (20 * 2 * \cos 37°)\ \text{J} = 32\ \text{J}$$

$$W_{net} = W_1 + W_2 = (100 + 32)\ \text{J} = 132\ \text{J}$$

---

b) calculate its speed at the end of the displacement.

   Solution: $m = 20$ kg; $v_i = 0$ m / s; $v_f = ?$

$$W_{net} = mv_f^2/2 - mv_i^2/2 = mv_f^2/2$$

$$v_f^{\,2} = 2W_{net}/m = 2 * 132/20 \text{ m}^2/\text{s}^2 = 13.2 \text{ m}^2/\text{s}^2$$

$$v_f = \sqrt{(13.2)} \text{ m/s} = 3.6 \text{ m/s}$$

5.2 Conservative Forces

Conservative forces are forces for which the work done is independent of the path followed. The work done depends only on a value on its initial location and final location that we call the potential energy of the object at the given location. *Potential Energy* of an object is the energy an object possesses just because of its location. For example an object separated from earth has potential energy just because of its location. If the object is released, this energy is converted to kinetic (motion) energy. The unit of measurement for potential energy is the Joule. The work done by a conservative force over a closed path is zero because the initial and final locations are identical. Examples of conservative forces are gravitational force, electrical force, magnetic force and the force due to a spring.

The work done by a conservative force is equal to the negative of the change of the potential energy associated with the force. Potential energy depends only on position coordinates. But the way the potential energy depends on position coordinates is different for different kinds of forces.

$$W_c = -\Delta PE_c = -(PE_{cf} - PE_{ci})$$

W_c is work done by a certain conservative force. PE_c is potential energy associated with the conservative force. PE_{ci} and PE_{cf} are the potential energies at the initial and final locations respectively.

Example: An object is moved from a location where the potential energy is *100* J to a location where the potential energy is *200* J. Calculate the work done by the conservative force associated with the potential energy.

Solution: $PE_{ci} = 100$ J; PE_{cf}; $W_c = ?$

$$W_c = -(PE_{cf} - PE_{ci}) = -(200 - 100) \text{ J} = -100 \text{ J}$$

5.2.1 Principle of Conservation of Mechanical Energy

The *mechanical energy* of an object is defined to be the sum of the kinetic energy and potential energy of the object.

$$ME = KE + PE$$

ME is the mechanical energy of an object whose kinetic and potential energies are *KE* and *PE* respectively. The expression for the potential energy varies from force to force. Also, sometimes there may be more than one conservative forces involved. On the other hand, the expression for kinetic energy is always $mv^2/2$.

$$ME = mv^2/2 + PE$$

Example: An object of mass *4* kg has a speed of *20* m/s when at a point where the object has a potential energy of *50* J. Calculate its mechanical energy.

Solution: $m = 4$ kg; $v = 20$ m/s; $PE = 50$ J; ME = ?

$$ME = mv^2/2 + PE = (4 * 20^2/2 + 50)\text{ J} = 850\text{ J}$$

If all the forces with a non-zero contribution to the net work done acting on an object are conservative, then the net work done on the object is equal to the work done by the conservative forces. According to the work kinetic energy, the net work done on an object is equal to the change of its kinetic energy. And the work done by conservative forces is equal to the negative of the change in its potential energy. Therefore, if all the forces acting on an object are conservative the following equation holds:

$$\Delta KE = -\Delta PE$$

But $\Delta KE = KE_f - KE_i$ and $\Delta PE = PE_f - PE_i$. After substituting these expressions and collecting the initials on one side and the finals on the other side, the following equation is obtained.

$$KE_i + PE_i = KE_f + PE_f$$

The left hand side of this equation is equal to the initial mechanical energy of the object and the right hand side of this equation is equal to the final mechanical energy of the object. The mechanical energy of the object remains the same. In other words the mechanical energy of the object is conserved. This is a mathematical statement of the principle of conservation of mechanical energy. The *principle of conservation of mechanical energy* states that if all the forces with none-zero contribution to the work done acting on an object are conservative, then the mechanical energy of the object is conserved.

Example: A *5* kg object was displaced from a location where its potential energy is *20* J to a location where its potential energy is *10* J under the influence of conservative forces only. If it's initial speed is *2* m/s,

a) Calculate its initial kinetic energy.

Solution: $m = 5$ kg; $v_i = 2$ m/s; $KE_i = ?$

$$KE_i = mv_i^2/2 = (5 * 2^2/2)\text{ J} = 10\text{ J}$$

---

b) calculate its initial mechanical energy.

Solution: $PE_i = 10$ J; $ME_i = ?$

$$ME_i = KE_i + PE_i = (10 + 20)\text{ J} = 30\text{ J}$$

c) Calculate its mechanical energy at its final location.

Solution: Since all the forces acting on the object are conservative, mechanical energy is conserved.

$$ME_f = ?$$

$$ME_f = ME_i = 30\text{ J}$$

d) calculate its kinetic energy at the end of the displacement.

Solution: $PE_f = 10$ J; $KE_f = ?$

$$ME_f = KE_f + PE_f$$

$$KE_f = ME_f - PE_f = (30 - 10)\text{ J} = 20\text{ J}$$

---

e) calculate its speed at the end of the displacement.

Solution: $v_f = ?$

$$KE_f = mv_f^2/2$$

$$v_f = \sqrt{(2\ KE_f/m)} = \sqrt{(2 * 20/5)}\ \mathrm{J} = 2.8\ \mathrm{m/s}$$

5.3 Practice Quiz 5.1

Choose the best answer. Answers can be found at the back of the book.

1. The work done by a force will be negative if
   A. if the angle formed between the force and the displacement is obtuse (i.e., between 90 and 180 degree)
   B. the force is opposite to the displacement only.
   C. if the angle formed between the force and the displacement is acute (i.e., between 0 and 90 degree)
   D. the force is parallel to the displacement only.
   E. the force is perpendicular to the displacement.

2. An object is displaced by a distance of *10* m horizontally by a *200* force that makes an angle of *20* deg with the horizontal. Calculate the work done by the force.
   A. *2819.078* J
   B. *375.877* J
   C. *939.693* J
   D. *3758.77* J
   E. *1879.385* J

3. An object is being pulled to the left by a *80* N force that makes an angle of *40* deg with the horizontal-left while moving to the right for a distance of *80* m. Calculate the work done by the force.
   A. *-4902.684* J
   B. *3922.148* J
   C. *-3922.148* J
   D. *-5883.221* J
   E. *4902.684* J

4. A 5 kg object is displaced to the right by a distance of *18* m under the influence of the following forces: a *30* N force pulling to the right, a *18* N force that makes an angle of *40* deg with the horizontal-left (west). Calculate the net work done on the object.
   A. *291.802* J
   B. *408.522* J
   C. *320.982* J
   D. *379.342* J
   E. *350.162* J

5. Calculate the kinetic energy of an object of mass *24* kg moving with a speed of *32* m / s.
   A. *14745.6* J
   B. *9830.4* J
   C. *12288* J
   D. *17203.2* J
   E. *7372.8* J

6. Under the influence of some forces, the speed of a *21* kg object changed from *25* m/s to *90* m/s. Calculate the net work done by the forces.
   A. *94185* J
   B. *78487.5* J
   C. *-78487.5* J
   D. *62790* J
   E. *-62790* J

7. Which of the following statements is correct.
   A. The work done by a conservative force is equal to the change in the kinetic energy of the object.
   B. The potential energy of an object depends on the speed of the object.
   C. The work done by a conservative force depends on the path followed.
   D. Friction is a conservative force.
   E. The work done by a conservative force is equal to the negative of the change in potential energy of the object.

8. An object is displaced *11* m by a *74* N conservative force that makes an angle of *20* deg with the displacement. Calculate the change in its potential energy.
   A. *-611.928* J
   B. *-535.437* J
   C. *-764.91* J
   D. *764.91* J
   E. *535.437* J

9. Calculate the mechanical energy of a *10.4* kg object at a location where its potential energy is *10* J if its speed as it crosses this location is *11* m/s.
   A. *447.44* J
   B. *383.52* J
   C. *767.04* J
   D. *639.2* J
   E. *575.28* J

10. Under the influence of conservative forces only, an object is displaced from point A to point B. Its potential energy and kinetic energy at point A are respectively *44* J and *50* J. Its kinetic energy at point B is *35* J. Calculate its potential energy at point B.
   A. *53.1* J
   B. *35.4* J
   C. *76.7* J
   D. *59* J
   E. *70.8* J

11. Under the influence of conservative forces only, an object of mass *9* kg is displaced from point A to point B. Its potential energy and speed at point A respectively are *1050* J and *24* m / s. Its potential energy at point B is *-300* J. Calculate its speed at point B.
    A. *23.678* m / s
    B. *20.718* m / s
    C. *17.758* m / s
    D. *32.557* m / s
    E. *29.597* m / s

5.3.1 Gravitational Potential Energy

Gravitational potential energy is potential energy associated with the conservative force gravitational force in such a way that the work done by gravitational force is equal to the negative of the change in gravitational potential energy.

$$W_g = -(PE_{gf} - PE_{gi}) = -\Delta PE_g$$

W_g is work done by gravitational force in moving an object from a certain initial location to a certain final location. PE_{gi} and PE_{gf} are gravitational potential energies at the initial and final locations of the object respectively. An expression for gravitational potential energy in terms of position coordinate(s) may be obtained by obtaining an expression for the work done by gravitational force in terms of position coordinate(s).

Suppose an object of mass m is displaced from the location (x_i, y_i) to the location (x_f, y_f) under the influence of gravitational force. The work done by gravitational force is independent of the path followed from the initial to the final location because gravitation force is a conservative force. We may pick the easiest path to obtain an expression for the work done by gravitational force during this displacement. Let's take the vertical path that takes from the initial location (x_i, y_i) to (x_i, y_f) and then the horizontal path that takes to the final location (x_f, y_f). Gravitational force (weight) is directed vertically down. The horizontal path need not be considered because over that path gravitational force and displacement are perpendicular to each other and the work done is zero. For the vertical path, without loss of generality, let's assume the final location to be below the initial location. In that case, gravitational force and the displacement are both directed vertically down and the angle between them is zero. The magnitude of the displacement is equal to the difference between the initial and final y coordinates, $y_i - y_f$. The magnitude of gravitational force is equal to the weight of the object which is equal to $m|g|$. Applying the definition for work, $W_g = F_g\, d \cos \theta$, the following expression can be obtained.

$$W_g = -m|g|(y_f - y_i)$$

Work done by gravity depends on the y-coordinates of the initial and final locations only. By comparing this equation with $W_g = -(PE_{gf} - PE_{gi})$, the gravitational potential energy at an arbitrary point whose y-coordinate is y is defined as follows:

$$PE_g = m|g|y$$

PE_g is the gravitational potential energy of an object of mass m located at a point whose y-coordinate is y. Potential energy depends on the choice of reference point because position does. An object may have different potential energies for different coordinate systems. But, customarily, the reference point (origin) is put on the ground which makes the y-coordinate synonymous with the height of the object. The mechanical energy of an object under the influence of gravitational force (ME_g) may now be expressed as

$$ME_g = mv^2/2 + m|g|y$$

Example: An object of mass *10* kg is located on top of a *2* m table.

a) Calculate its gravitational potential energy with respect to the ground.

Solution: With respect to the ground means the reference point or the origin of the coordinate system should be fixed at the ground.

$m = 10$ kg; $y = 2$ m; $PE_g = ?$

$$PE_g = m|g|y = (10 * 9.8 * 2) \text{ J} = 196 \text{ J}$$

b) Calculate its gravitational potential energy with respect to the top of the table.

Solution: The origin of the coordinate system is fixed at the top of the table.

$y = 0$ m; $PE_g = ?$

$$PE_g = m|g|y = (10 * 9.8 * 0) \text{ J} = 0 \text{ J}$$

c) Calculate its gravitational potential energy with respect to a point *5* m above the top of the table.

Solution: The origin is *5* m above the location of the object.

$y = -5$ m; $PE_g = ?$

$$PE_g = m|g|y = (10 * 9.8 * -5) \text{ J} = -490 \text{ J}$$

d) Calculate the mechanical energy of the object with respect to the ground if it is sliding on the table with a speed of *2* m/s.

Solution: $y = 2$ m; $v = 2$ m/s; $ME_g = ?$

$$ME_g = mv^2/2 + m|g|y = (10 * 2^2/2 + 10 * 9.8 * 2) \text{ J} = 216 \text{ J}$$

Example: An object of mass *20* kg is dropped from a height of *50* m. Calculate the work done by gravitational force.

Solution: Let the origin of the coordinate system be fixed at the ground.

$m = 20$ kg; $y_i = 50$ m; $y_f = 0$ m; $W_g = ?$

$$W_g = -m|g|(y_f - y_i) = -20 * 9.8 * (0 - 50) \text{ J} = 9800 \text{ J}$$

Example: An object of mass *10* kg is displaced from the location *(2, 4)* m to the location *(10, 34)* m. Calculate the work done by gravitational force.

Solution: $(x_i, y_i) = (2, 4)$ m. $(x_f, y_f) = (10, 34)$ m.

$m = 10$ kg; $y_i = 4$ m; $y_f = 34$ m; $W_g = ?$

$$W_g = -m|g|(y_f - y_i) = -10 * 9.8 * (34 - 4) \text{ J} = -29400 \text{ J}$$

5.3.1 Conservation of Mechanical Energy for Gravitational Force

That now we have an expression for gravitational potential energy, we can specialize the general expression for conservation of mechanical energy to gravitational forces.

$$mv_i^2/2 + m|g|y_i = mv_f^2/2 + m|g|y_f$$

Example: An object is dropped from a height of *10* m. Use the principle of conservation of mechanical energy to calculate its speed by the time it hits the ground.

Solution: Let the origin of the coordinate system be fixed at the ground. Its initial speed is zero because it is released from rest.

$y_i = 10$ m; $y_f = 0$ m; $v_i = 0$ m/s; $v_f = ?$

$$mv_i^2/2 + m|g|y_i = mv_f^2/2 + m|g|y_f$$

$$m|g|y_i = mv_f^2/2$$

$$v_f = \sqrt{(2|g|y_i)} = \sqrt{(2 * 9.8 * 10)} \text{ m/s} = 14 \text{ m/s}$$

Example: An object is sliding down a frictionless *10* m *53°* inclined plane. Calculate its speed by the time it reaches the ground.

Solution: The forces acting on the object are gravitational force and normal force exerted by the surface of the plane. The normal force is perpendicular to the displacement and thus the work done by the normal force does not contribute to the work done. Since the only force with non-zero contribution to the work done (gravity) is conservative, the principle of conservation of mechanical energy can be used.

Let the origin of the coordinate system be fixed at the final location (where it hits the ground) of the object. Then, $(x_f, y_f) = (0, 0)$ m and $(x_i, y_i) = (10 * \cos 53°, 10 * \sin 53°)$ m $= (6, 8)$ m. Its initial speed is zero because it is just sliding.

$y_i = 8$ m; $y_f = 0$ m; $v_i = 0$; $v_f = ?$

$$mv_i^2/2 + m|g|y_i = mv_f^2/2 + m|g|y_f$$

$$m|g|y_i = mv_f^2/2$$

$$v_f = \sqrt{(2|g|y_i)} = \sqrt{(2 * 9.8 * 8)} \text{ m/s} = 11.2 \text{ m/s}$$

Example: An object is released from a frictionless roller coaster of height *30* m with an initial speed of *5* m ⁄ s. The roller coaster drops to the ground level and then rises to a height of *20* m and then falls to the ground level again. Calculate the speed of the object by the time it is at the top of the *20* m loop.

Solution: The forces acting on the object are gravitational force and the normal force exerted by the roller coaster. The normal force does not contribute to the work done because it is perpendicular to the displacement. Only gravity contributes to the work done and hence the principle of conservation of mechanical energy can be used. Let the origin of the coordinate system be fixed at the ground.

$y_i = 30$ m; $y_f = 20$ m; $v_i = 5$ m/s; $v_f = ?$

$$mv_i^2/2 + m|g|y_i = mv_f^2/2 + m|g|y_f$$

$$v_f^2 = v_i^2 + 2|g|(y_i - y) = \{5^2 + 2 * 9.8 *(30 - 20)\} \text{ m}^2/\text{s}^2 = 221 \text{ m}^2/\text{s}^2$$

$$v_f = \sqrt{(221)} \text{ m/s} = 14.9 \text{ m/s}$$

5.3.2 Elastic Potential Energy

Elastic potential energy is potential energy associated with the force due to a spring such that the work done by a force due to a spring is equal to the negative of the change of elastic potential energy of the spring.

$$W_s = -(PE_{sf} - PE_{si}) = -\Delta PE_s$$

W_s is the work done by the force of a spring. PE_{si} and PE_{sf} are the potential energies of the spring at the initial and final extensions (compressions) of the spring. The force due to a spring is related to the extension (compression) of the spring by a law known as Hook's Law. *Hook's Law* states that the force due to a spring is proportional but opposite in direction to the displacement (extension or compression) of the spring.

$$F_{sx} = -kx$$

F_{sx} is force exerted by a spring. x is the displacement of the spring as measured from the relaxed position of the spring. It is positive for extension and negative for compression. k is a proportionality constant for a given spring and is called Hook's constant of the spring. Its unit of measurement is N / m. k is always positive. This equation is a relationship between components. The force and the displacement can be positive or negative. The negative in the equation indicates that the force and the displacement have opposite signs or directions. If a relationship between the magnitudes is desired, the magnitudes of both sides should be equated.

$$F_s = k|x|$$

Example: A certain spring extends by 0.1 m when subjected to a 10 N force.

a) Calculate the Hook's constant of the spring.

Solution: $F_s = 10$ N; $|x| = 0.1$ m; $k = ?$

$$F_s = k|x|$$

$$k = F_s / |x| = (10 / 0.1) \text{ N/m} = 100 \text{ N/m}$$

---

b) What force would extend it by *0.35* m

Solution: $|x| = 0.35$ m; $F_s = ?$

$$F_s = k|x| = (100 * 0.35) \text{ N} = 35 \text{ m}$$

An expression for the elastic potential energy of a spring may be obtained by first obtaining an expression for the work done by the force due to a spring. The force due to a spring varies as a function of the displacement. The formula for work done being used so far ($W = Fd \cos \theta$) cannot be used in this case because it applies only if the force remains constant during the displacement. The work done by a variable force may be obtained from the graph of force versus displacement as the area enclosed between the force versus displacement curve and the displacement axis. Areas above the displacement axis are taken to be positive and areas below the displacement axis are taken to be negative. Since $F_{sx} = -kx$, the graph of force due to a spring versus displacement (extension) is a straight line in the fourth quadrant. The work done in extending a spring from its relaxed position to an arbitrary extension x is equal to the area of a triangle of base x and height kx and is negative which is equal to $-kx^2$. The work done in displacing a spring from an initial position x_i to a final position x_f may be obtained from the difference between the areas of the triangles associated with both triangles.

$$W_s = -(kx_f^2/2 - kx_i^2/2)$$

Comparing this expression with the expression $W_s = -(PE_{sf} - PE_{si})$, the elastic potential energy (PE_s) of a spring with arbitrary extension or compression x is defined as follows:

$$PE_s = kx^2/2$$

The mechanical energy (ME_s) of an object attached to a spring is given by

$$ME_s = mv^2/2 + kx^2/2$$

Example: Calculate the elastic potential energy stored by a spring of Hook's constant *200* N/m when extended by *0.04* m.

Solution: $k = 200$ N/m; $x = 0.04$ m; $PE_s = ?$

$$PE_s = kx^2/2 = 200 * 0.04^2/2 \text{ J} = 0.16 \text{ J}$$

5.3.3 Conservation of Mechanical Energy for a Force due to a Spring

The principle of conservation of mechanical energy may be specifically written for a spring by using the expression for elastic potential energy as a function of position.

$$mv_i^2/2 + kx_i^2/2 = mv_f^2/2 + kx_f^2/2$$

Example: An object of mass *2* kg is attached to a spring of Hook's constant *100* N / m. The spring is compressed by *0.01* m and then released on a horizontal frictionless surface. Calculate the speed of the object when it is at the relaxed position of the spring.

Solution: $m = 2$ kg; $k = 100$ N / m; $x_i = -0.01$ m; $x_f = 0$ m (relaxed position); $v_i = 0$ m / s (released from rest); $v_f = ?$

$$mv_i^2/2 + kx_i^2/2 = mv_f^2/2 + kx_f^2/2$$

$$kx_i^2 = mv_f^2$$

$$v_f = \sqrt{(k/m)}|x_i| = \sqrt{(100/2)} * 0.01 \text{ m/s} = 0.071 \text{ m/s}$$

5.4 Work done by Non-Conservative forces

Non-conservative forces are forces for which the work done depends on the path followed. Typical examples are friction and air resistance. The forces acting on an object may be classified into conservative and non-conservative forces.

$$W_{net} = W_c + W_{nc}$$

W_{net} is the net work done on the object. W_c and W_{nc} are the work done by the conservative and non-conservative forces respectively. From the work kinetic energy theorem, net work done is equal to the change in the kinetic energy of the object and work done by a conservative force is equal to the negative of the change in potential energy. Therefore

$$W_{nc} = \Delta KE + \Delta PE = \Delta ME$$

Work done by a non-conservative force is equal to the change (loss) in the mechanical energy of the object. The effect of work done by a non-conservative force is to decrease the mechanical energy of the object. If gravity is the only conservative force acting on an object, then

$$W_{nc} = (mv_f^2/2 + m|g|y_f) - (mv_i^2/2 + m|g|y_i)$$

And if force due to a spring is the only conservative force acting on an object

$$W_{nc} = (mv_f^2/2 + kx_f^2/2) - (mv_i^2/2 + kx_i^2/2)$$

Example: An object of mass 0.02 kg is dropped from a height of 20 m. By the time it hits the ground, its speed is 10 m/s. Calculate the work done by air resistance.

Solution: The forces acting on the object are gravity and air resistance. Air resistance is the non-conservative force. Let the coordinate system be fixed at the ground.

$m = 0.02$ kg; $y_i = 20$ m; $y_i = 0$ m/s; $v_f = 10$ m/s; $W_{nc} = ?$

$$W_{nc} = (mv_f^2/2 + m|g|y_f) - (mv_i^2/2 + m|g|y_i) = \{(0.02 * 10^2/2 + 0) - (0 + 0.02 * 9.8 * 20)\} \text{ J} = -2.92 \text{ J}$$

Example: A car of mass 10^4 kg initially moving with a speed of 20 m/s on a horizontal surface was stopped by the force of friction. Calculate the work done by the force of friction.

Solution: The forces acting on the object are gravity and friction. Friction is the only non-conservative force acting on the object. Let the coordinate system be fixed on the ground.

$m = 10^4$ kg; $y_i = y_f = 0$ m; $v_i = 20$ m / s; $v_f = 0$ m / s; $W_{nc} = ?$

$$W_{nc} = (mv_f^2/2 + m|g|y_f) - (mv_i^2/2 + m|g|y_i) = \{ (0 + 0) - (10^4 * 20^2/2 + 0) \} \text{ J} = -2 * 10^6 \text{ J}$$

5.5 Practice Quiz 5.2

Choose the best answer. Answers can be found at the back of the book.

1. An object of mass *12* kg is located at the top of a *14* m tall building. Calculate the gravitational potential energy of the object with respect to a point *4* m below the top of the building.
   A. *517.44* J
   B. *282.24* J
   C. *564.48* J
   D. *470.4* J
   E. *611.52* J

2. A bullet of mass *0.05* kg is fired from a *7* m tall building horizontally with a speed of *700* m / s. Calculate its mechanical energy just after it is fired.
   A. *13478.773* J
   B. *17154.802* J
   C. *14704.116* J
   D. *8577.401* J
   E. *12253.43* J

3. An object of mass *3* kg is sliding down a friction less inclined plane of length *8* m that makes an angle of *20* deg with the horizontal. Calculate the work done by gravitational force as the object slides from the top of the inclined plane to the ground.
   A. *80.443* J
   B. *112.62* J
   C. *104.576* J
   D. *72.399* J
   E. *64.355* J

4. An object of mass *3* kg is sliding down a friction less inclined plane of length *8* m that makes an angle of *60* deg. Calculate its speed just before it hits the ground.
   A. *8.157* m / s
   B. *9.322* m / s
   C. *11.653* m / s
   D. *10.488* m / s
   E. *16.314* m / s

5. A roller coaster extends to the ground from a height of *32* m (point A) and then rises to a height of *11* m (point B). An object of mass *14* kg starts at point A with a speed of *6* m / s. If the roller coaster is friction less, calculate the speed of the object by the time it reaches point B.
   A. *24.388* m / s
   B. *12.694* m / s
   C. *27.504* m / s
   D. *21.157* m / s
   E. *19.041* m / s

6. A certain spring extends by *0.45* m when an object of mass *0.9* kg hangs from it. Calculate the Hook's constant of the spring.
   A. *27.44* N / m
   B. *19.6* N / m
   C. *17.64* N / m
   D. *25.48* N / m
   E. *15.68* N / m

7. Calculate the elastic potential stored by a spring of Hook's constant *200* N / m when it is compressed by *.225* m.
   A. *3.038* J
   B. *5.569* J
   C. *6.581* J
   D. *4.05* J
   E. *5.063* J

8. An object of mass *1* kg is brought in contact with a spring of Hook's constant *140* N / m that is compressed by *0.15* m. If the spring is let go free to expand, calculate the speed by which the object will leave the spring at its relaxed position.
   A. *1.42* m / s
   B. *1.242* m / s
   C. *1.775* m / s
   D. *1.952* m / s
   E. *2.485* m / s

9. Work done by the non-conservative forces acting on an object is equal
   A. to the change in the mechanical energy of the object
   B. to the net work done on the object
   C. to the work done by the conservative forces
   D. to the change in the potential energy of the object
   E. to the change in the kinetic energy of the object

10. Initially an object has a potential energy of *30* J and a kinetic energy of *200* J. Under the influence of conservative and non-conservative forces its potential energy changed to *30* J and its kinetic energy changed to *26* J. Calculate the work done by the non-conservative forces.
    A. -*166* J
    B. -*176* J
    C. -*174* J
    D. -*181* J
    E. -*173* J

11. An object of mass *0.03* kg released from a height of *11* m has a speed of *5* m/s just before it hits the ground. Calculate the work done by air resistance.
    A. -*2.001* J
    B. -*2.859* J
    C. -*3.431* J
    D. -*2.573* J
    E. -*1.715* J

6 Momentum and Collisions

Your goals for this chapter are to learn about momentum, impulse, principle of conservation of momentum and collisions.

Momentum is a physical quantity used as a measure of the amount of motion an object has. It is directly proportional to the mass and the velocity of the object. It is defined to be the product of the mass and velocity of the object. Since velocity is a vector quantity, momentum is also a vector quantity.

$$\mathbf{p} = m\mathbf{v}$$

p is the momentum of an object of mass m moving with a velocity **v**. This is a vector equation. To obtain a relationship between the components along a certain direction, components of both sides along that direction should be taken.

$$p = mv$$

p and v are the components of the momentum and the velocity vectors respectively. The unit of measurement for momentum is kg m ⁄ s.

Example: Calculate the momentum of a 6 kg object moving with a speed of 4 m ⁄ s.

Solution: $m = 6$ kg; $v = 4$ m ⁄ s; $p = ?$

$$p = mv = 6 * 4 \text{ kg m/s} = 24 \text{ kg m/s}$$

Impulse of an object is defined to be the product of the force acting on the object and the time interval during which the force is applied. Impulse is a vector quantity. The unit of measurement for impulse is N s.

$$\mathbf{I} = \mathbf{F}\Delta t$$

I is impulse of an object acted upon by a force **F** for a time interval of Δt. This is a vector equation. A component equation may be obtained by equating the components of both sides.

$$I = F\Delta t$$

Example: An object was acted upon by a force of *50* N for *0.2* seconds. Calculate the impulse acting on the object.

Solution: $F = 50$ N; $\Delta t = 0.2$ s; $I = ?$

$$I = F\Delta t = 50 * 0.2 \text{ N s} = 10 \text{ N s}$$

Impulse can be obtained from a graph of force versus time as the area enclosed between the force versus time curve and the time axis.

6.1 Principle of Conservation of Momentum

According to Newton's second law, force is equal to the product of mass and acceleration and acceleration is equal to the rate of change of velocity with time.

$$F = m\Delta v / \Delta t = (mv_f - mv_i) / \Delta t$$

But $mv_i = p_i$ and $mv_f = p_f$.

$$F = (p_f - p_i) / \Delta t = \Delta p / \Delta t$$

The *relationship between force and momentum* is that force acting on an object is equal to the rate of change of its momentum with time. This relationship also implies that the change in the momentum of an object is equal to the product of force and interval of time which is in turn equal to the impulse acting on the object.

$$I = \Delta p$$

Example: A ball of mass *0.1* kg moving to the right hits a wall with a speed of *10* m/and bounces back with a speed of *6* m/s.

a) Calculate the change in its momentum.

   Solution: Remember a component to the right is taken to be positive and a component to the left is taken to be negative.

   $m = 0.1$ kg; $v_i = 10$ m/s; $v_f = -6$ m/s; $\Delta p = ?$

$$\Delta p = p_f - p_i = mv_f - mv_i = (0.1 * -6 - 0.1 * 10) \text{ kg m/s} = -1.6 \text{ kg m/s}$$

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b) Calculate the impulse impacted on the object during the collision.

Solution: $I = ?$

$$I = \Delta p = -1.6 \text{ N s}$$

c) If the ball was in contact with the wall for 0.2 seconds, calculate the average force exerted by the wall on the ball.

Solution: $\Delta t = 0.2$ s; $F = ?$

$$F = \Delta p / \Delta t = -1.6/0.2 \text{ N s} = -8 \text{ N s}$$

The negative means the direction of the force is to the left.

If the net force acting on an object is zero, then $\Delta p = F\Delta t = 0$. That is the momentum of the object does not change. In other words the momentum of the object is conserved. The *principle of conservation of momentum* states that if the net force acting on an object is zero, then the momentum of the object is conserved.

If

$$F = 0$$

then

$$p_f = p_i$$

6.2 Conservation of Momentum for Collisions

When two objects collide, each object exerts force on the other. These two forces are action reaction forces. They are opposites of each other. If the two objects are treated as a single system, these two forces cancel each other and the net force acting on the system during the collision is zero. And from the principle of conservation of momentum, it follows that momentum of colliding objects is conserved if they are treated as a single system.

6.2.1 One Dimensional Collision

One dimensional collision is collision in a straight line. Suppose an object of mass m_1 moving with a speed v_{1i} collides with an object of mass m_2 moving with a speed of v_{2i}. And suppose after collision, the former moves with a speed of v_{1f} and the later moves with a speed of v_{2f}. Since momentum of colliding objects is conserved, the momentum of these colliding objects before collision is equal to their momentum after collision.

$$m_1v_{1i} + m_2v_{2i} = m_1v_{1f} + m_2v_{2f}$$

Example: A *10* kg object moving to the right with a speed of *5* m/s collides with a *2* kg object moving to the left with a speed of *3* m/s. After collision, the *2* kg object moves to the right with a speed of *10* m/s. Calculate the speed of the *10* kg object after collision.

Solution: $m_1 = 10$ kg; $v_{1i} = 5$ m/s; $m_2 = 2$ kg; $v_{2i} = -3$ m/s; $v_{2f} = 10$ m/s; $v_{1f} = ?$

$$m_1v_{1i} + m_2v_{2i} = m_1v_{1f} + m_2v_{2f}$$

$$m_1v_{1f} = m_1v_{1i} + m_2v_{2i} - m_2v_{2f} = (10 * 5 + 2 * -3 - 2 * 10) \text{ kg m/s} = 24 \text{ kg m/s}$$

$$v_{1f} = 24/10 \text{ m/s} = 2.4 \text{ m/s}$$

6.2.2 Completely Inelastic Collision

A *Completely inelastic collision* is a collision where the two objects stick together after collision. Both objects will have the same speed after collision: $v_{1f} = v_{2f} = V$.

$$m_1v_{1i} + m_2v_{2i} = (m_1 + m_2)V$$

V is their common speed after collision. Kinetic energy is not conserved during inelastic collision. Some of the initial kinetic energy is lost as heat energy during the collision.

Example: A 7 kg object moving to the right with a speed of 8 m/s collides with a 3 kg object moving to the left with a speed of 5 m/s. If the collision is completely inelastic

a) Calculate their speed after the collision.

Solution: $m_1 = 7$ kg; $v_{1i} = 8$ m/s; $m_2 = 3$ kg; $v_{2i} = -5$ m/s; $V = ?$

$$m_1v_{1i} + m_2v_{2i} = (m_1 + m_2)V$$

$$V = (m_1v_{1i} + m_2v_{2i})/(m_1 + m_2) = (7 * 8 + 3 * -5)/(7 + 3) \text{ m/s} = 4.1 \text{ m/s}$$

---

b) Calculate the kinetic energy lost during the collision.

Solution: $\Delta KE = KE_f - KE_i = ?$

$$\Delta KE = (m_1 + m_2)V^2/2) - (m_1v_{1i}^2/2 + m_2v_{2i}^2/2)$$

$$= \{ (7 + 3)* 4.1^2/2) - (7 * 8^2// 2 - 3 * -5^2/2) \} \text{ J}$$

$$= -177.4 \text{ J}$$

6.3 Practice Quiz 6.1

Choose the best answer. Answers can be found at the back of the book.

1. The unit of measurement for momentum is
   A. kilogram * meter/second
   B. kilogram * meter/second ^{2}
   C. kilogram * meter * second
   D. Joule
   E. Newton/second

2. Calculate the momentum of an object of mass *0.08* kg moving with a speed of *5.4* m / s.
   A. *0.432* kg m / s
   B. *0.389* kg m / s
   C. *0.346* kg m / s
   D. *0.302* kg m / s
   E. *0.605* kg m / s

3. A object of mass *0.2* kg moving to the right with a speed of *15* m / s hits a wall and bounces back to the left with a speed of *4* m / s. Calculate the change in its momentum.
   A. -*3.8* kg m / s
   B. -*5.32* kg m / s
   C. -*3.04* kg m / s
   D. -*4.94* kg m / s
   E. -*3.42* kg m / s

4. An object of mass *0.32* kg falling vertically downward hits the ground with a speed of *15* m / s and bounces back vertically upward with a speed *8* m / s. If the object was in contact with the ground for *0.5* seconds, calculate the average force exerted by the ground on the object.
   A. *17.664* N
   B. *20.608* N
   C. *10.304* N
   D. *14.72* N
   E. *16.192* N

5. An object was acted upon by a force of *24* for *0.6* seconds. Calculate the change in its momentum.
   A. *14.4* kg m / s
   B. *10.08* kg m / s
   C. *11.52* kg m / s
   D. *8.64* kg m / s
   E. *12.96* kg m / s

6. An object of mass *9.5* kg was acted by a force of *30.2* N for *5.4* seconds. If its initial speed was *3.2* m / s, calculate its final speed.
   A. *16.293* m / s
   B. *26.476* m / s
   C. *20.366* m / s
   D. *18.33* m / s
   E. *28.513* m / s

7. An object of mass *20* kg moving with a speed of *28* m / s to the right collides with an object of mass *19* kg moving with a speed of *10* m / s in the same direction. After collision, the *20* kg object moves to the right with a speed of *5* m / s to the right. Calculate the velocity of the *19* kg object after collision.
   A. *34.211* m / s
   B. *20.526* m / s
   C. *41.053* m / s
   D. *37.632* m / s
   E. *23.947* m / s

8. An object of mass *14* kg moving with a speed of *25* m / s to the right collides with an object of mass *13* kg moving with a speed of *14* m / s to the left. After collision, the *13* kg object moves to the right with a speed of *2* m / s to the right. Calculate the velocity of the *14* kg object after collision.
   A. *8.114* m / s
   B. *9.129* m / s
   C. *6.086* m / s
   D. *10.143* m / s
   E. *13.186* m / s

9. An object of mass *10* kg moving with a speed of *28* m / s to the right collides with an object of mass *17* kg moving with a speed of *18* m / s to the left. If the collision is completely inelastic, calculate their speed after collision.
   A. *3.393* m / s
   B. *0.16* m / s
   C. *-0.963* m / s
   D. *1.271* m / s
   E. *2.694* m / s

10. An object of mass *2* kg moving with a speed of *22* m / s to the right collides with an object of mass *17* kg at rest. If the collision is completely inelastic, calculate the kinetic energy lost during the collision.
    A. *-519.663* J
    B. *-562.968* J
    C. *-606.274* J
    D. *-476.358* J
    E. *-433.053* J

6.3.1 The Ballistic Pendulum

The *ballistic pendulum* is a pendulum used to measure the speed of a bullet. A bullet is fired into the pendulum and gets embedded in the pendulum. As a result the pendulum rises to a certain height. If the masses of the pendulum and the bullet are known, the speed of the bullet can be calculated using this height.

Let the mass of the bullet be m_b, the speed of the bullet be v_b, the mass of the pendulum be m_p, their common speed after the collision be V and the height to which the pendulum rises after the collision be h. The collision is completely inelastic and the initial speed of the pendulum is zero. Therefore

$$m_b v_b = (m_b + m_p)V$$

And

$$v_b = (m_b + m_p)V/m_b$$

V may be expressed in terms of h by using the principle of conservation of mechanical energy. While the pendulum rises up, there are two forces acting on the pendulum: The tension in the string and gravitational force. The tension in the string does not contribute to the work done because it is perpendicular to the displacement. The only force with non-zero contribution to the work done is gravitational force which is conservative. Hence mechanical energy of the pendulum is conserved. Its mechanical energy at the bottom and at its maximum height are equal. Assuming the origin of the coordinate system is fixed at its initial location, $y_i = 0$, $y_f = h$, $v_i = V$ and $v_f = 0$.

$$(m_b + m_p)V^2 = (m_b + m_p)|g|h$$

And

$$V = \sqrt{(2|g|h)}$$

Substituting the V in the expression for v_b with this expression, the following expression for the speed of the bullet is obtained in terms of the masses of the bullet and the pendulum and the height to which the bullet rises after the collision.

$$v_b = (m_b + m_p)\sqrt{(2|g|h)} / m_b$$

Example: A bullet of mass *0.05* kg is fired into a pendulum of mass *10* kg. The bullet is embedded in the pendulum and the pendulum rises to a height of *2* m. Calculate the speed of the bullet.

Solution: $m_b = 0.05$ kg; $m_p = 10$ kg; $h = 2$ m; $v_b = ?$

$$v_b = (m_b + m_p)\sqrt{(2|g|h)} / m_b = \{ (0.05 + 10)\sqrt{(2 * 9.8 * 2)} / 0.05 \} \text{ m/s} = 1258.5 \text{ m/s}$$

6.3.2 Completely Elastic Collision

A *completely elastic collision* is a collision where not only momentum but also kinetic energy is conserved. For a completely elastic collision, the following two equations from the conservation of momentum and kinetic energy hold.

$$m_1v_{1i} + m_2v_{2i} = m_1v_{1f} + m_2v_{2f}$$

$$m_1v_{1i}^2/2 + m_2v_{2i}^2/2 = m_1v_{1f}^2/2 + m_2v_{2f}^2/2$$

The second equation is not in a desirable form because it contains velocities raised to the power of *2*. It can be simplified though with little mathematical manipulation. Collecting terms containing m_1 on one side and terms containing m_2 on the other side, (and factorizing the resulting differences between squares for the second equation) the following equations are obtained.

$$m_1 (v_{1i} - v_{1f}) = m_2 (v_{2f} - v_{2i})$$

$$m_1 (v_{1i} - v_{1f})(v_{1i} + v_{1f}) = m_2 (v_{2f} - v_{2i})(v_{2f} + v_{2i})$$

After dividing the second equation by the first equation and collecting the initial velocities of the resulting equation on one side and the final velocities on the other side, the following equation is obtained.

$$v_{1i} - v_{2i} = -(v_{1f} - v_{2f})$$

Using this simplified equation, the following two equations can be used for a completely elastic collision.

$$m_1v_{1i} + m_2v_{2i} = m_1v_{1f} + m_2v_{2f}$$

$$v_{1i} - v_{2i} = -(v_{1f} - v_{2f})$$

Example: A 4 kg object moving to the right with a speed of 5 m / s collides with a 2 kg object moving to the right with a speed of 3 m / s. If the collision is completely elastic, calculate the speeds of both objects after collision.

Solution: $m_1 = 4$ kg; $v_{i1} = 5$ m / s; $m_2 = 2$ kg; $v_{i2} = 3$ m / s; $v_{1f} = ?$; $v_{2f} = ?$

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$

$$m_1 v_{1f} = m_1 v_{1i} + m_2 v_{2i} - m_2 v_{2f} = (4 * 5 + 2 * 3)\ \text{kg m / s} - 2\ \text{kg}\ v_{2f} = 26\ \text{kg m / s} - 2\ \text{kg}\ v_{2f}$$

$$v_{1f} = (26\ \text{kg m / s} - 2\ \text{kg}\ v_{2f}) / m_1 = (26\ \text{kg m / s} - 2\ \text{kg}\ v_{2f}) / 4\ \text{kg} = 6.5\ \text{m / s} - 0.5 v_{2f}$$

$$v_{1i} - v_{2i} = -(v_{1f} - v_{2f})$$

$$v_{2f} = v_{1i} - v_{2i} + v_{1f} = (5 - 3)\ \text{m / s} + v_{1f} = 2\ \text{m / s} + v_{1f}$$

And substituting the expression for v_{1f} in terms of v_{2f}

$$v_{2f} = 2\ \text{m / s} + 6.5\ \text{m / s} - 0.5 v_{2f}$$

$$1.5 v_{2f} = 8.5\ \text{m / s}$$

$$v_{2f} = 8.5 / 1.5\ \text{m / s} = 5.7\ \text{m / s}$$

And

$$v_{1f} = 6.5\ \text{m / s} - 0.5 v_{2f} = (6.5 - 0.5 * 5.7)\ \text{m / s} = 3.7\ \text{m / s}$$

6.3.3 Two Dimensional Collisions

Two dimensional collision is collision in a plane where more than one different directions for the velocities are involved. The principle of conservation of momentum applies in vector form.

$$m_1 \mathbf{v}_{1i} + m_2 \mathbf{v}_{2i} = m_1 \mathbf{v}_{1f} + m_2 \mathbf{v}_{2f}$$

If two vectors are equal, then their components also must be equal. Thus, this vector equation can be decomposed into two algebraic component equations: an equation for the x-components and an equation for the y-components.

$$m_1 v_{1ix} + m_2 v_{2ix} = m_1 v_{1fx} + m_2 v_{2fx}$$

$$m_1 v_{1iy} + m_2 v_{2iy} = m_1 v_{1fy} + m_2 v_{2fy}$$

Quite often, a velocity is given in the form of magnitude ($|v|$) and direction (θ). In that case, first the components should be calculated. If θ is a default angle (angle measured with respect to the positive x-axis), then $v_x = |v| \cos\theta$ and $v_y = |v| \sin\theta$.

Example: An object of mass 6 kg going north with a speed of 10 m/s collides with a 2 kg object going east with a speed of 20 m/s. After collision the 6 kg object moves with a speed of 15 m/s making an angle of 40° with the horizontal (east).

a) Calculate the components of the velocity of the 2 kg object after collision.

Solution: $m_1 = 6$ kg; $|v_{1i}| = 10$ m/s; $\theta_{1i} = 90°$; $|v_{1f}| = 15$ m/s; $\theta_{1f} = 40°$; $m_2 = 2$ kg; $|v_{2i}| = 20$ m/s; $\theta_{2i} = 0°$; $v_{2fx} = ?$; $v_{2fy} = ?$

$$v_{1ix} = |v_{1i}| \cos\theta_{1i} = (10 * \cos 90°)\ \text{m/s} = 0$$

$$v_{1iy} = |v_{1i}| \sin\theta_{1i} = (10 * \sin 90°)\ \text{m/s} = 10\ \text{m/s}$$

$$v_{2ix} = |v_{2i}| \cos\theta_{2i} = (20 * \cos 0°)\ \text{m/s} = 20\ \text{m/s}$$

$$v_{2iy} = |v_{2i}| \sin\theta_{2i} = (20 * \sin 0°)\ \text{m/s} = 0$$

$$v_{1fx} = |v_{1f}| \cos \theta_{1f} = (15 * \cos 40°) \text{ m/s} = 11.5 \text{ m/s}$$

$$v_{1fy} = |v_{1f}| \sin \theta_{1f} = (15 * \sin 40°) \text{ m/s} = 9.6 \text{ m/s}$$

$$m_1v_{1ix} + m_2v_{2ix} = m_1v_{1fx} + m_2v_{2fx}$$

$$m_2v_{2fx} = m_1v_{1ix} + m_2v_{2ix} - m_1v_{1fx} = (6 * 0 + 2 * 20 - 6 * 11.5) \text{ kg m/s} = -29 \text{ kg m/s}$$

$$v_{2fx} = -29/m_2 \text{ kg m/s} = -29/2 \text{ m/s} = -14.5 \text{ m/s}$$

$$m_1v_{1iy} + m_2v_{2iy} = m_1v_{1fy} + m_2v_{2fy}$$

$$m_2v_{2fy} = m_1v_{1iy} + m_2v_{2iy} - m_1v_{1fy} = (6 * 10 + 2 * 0 - 6 * 9.6) \text{ kg m/s} = 57.6 \text{ kg m/s}$$

$$v_{2fx} = 57.6/m_2 \text{ kg m/s} = 57.6/2 \text{ m/s} = 28.8 \text{ m/s}$$

---

b) Calculate the magnitude and the direction of the velocity of the *2* kg object after collision.

Solution: $|v_{2f}| = ?$; $\theta_{2f} = ?$

$$|v_{2f}| = \sqrt{(v_{2fx}^2 + v_{2fy}^2)} = \sqrt{\{(-14.5)^2 + 28.8^2\}} = 32.2 \text{ m/s}$$

$$\theta_{2f} = \arctan (v_{2fy}/v_{2fx}) + 180° = \arctan (28.8/-14.5) + 180° = 116.7°$$

6.4 Practice Quiz 6.2

Choose the best answer. Answers can be found at the back of the book.

1. A collision is said to be a completely elastic collision if,
   A. the two objects stick together after collision
   B. the momentum of the colliding objects is conserved during the collision.
   C. if the speeds of the objects after collision remain the same with the speeds before collision
   D. the kinetic energy of the colliding objects is conserved during the collision.
   E. the mechanical energy of the colliding objects is conserved during the collision.

2. Calculate the kinetic energy of an object of mass *20* kg if its momentum is *20* kg m/s.
   A. *9* J
   B. *7* J
   C. *10* J
   D. *14* J
   E. *12* J

3. After a bullet of mass *0.065* kg is fired into a ballistic pendulum of mass *0.5* kg, the bullet is embedded in the pendulum and the pendulum rose to a height of *0.045* m. Calculate the speed of the pendulum just after the collision.
   A. *1.127* m / s
   B. *0.563* m / s
   C. *0.845* m / s
   D. *0.657* m / s
   E. *0.939* m / s

4. After a bullet of mass *0.035* kg is fired into a ballistic pendulum of mass *1* kg, the bullet is embedded in the pendulum and the pendulum rose to a height of *0.2* m. Calculate the speed with which the bullet was fired into the ballistic pendulum.
   A. *64.403* m / s
   B. *70.258* m / s
   C. *40.984* m / s
   D. *81.968* m / s
   E. *58.548* m / s

5. An object of mass *17* kg going to the right with a speed of *15* m / s collides with a(n) *12* kg object at rest. If the collision is completely elastic, calculate the speed of the *12* kg object after collision.
   A. *22.862* m / s
   B. *21.103* m / s
   C. *17.586* m / s
   D. *10.552* m / s
   E. *15.828* m / s

6. An object of mass *17* kg going to the right with a speed of *27* m / s collides with a(n) *20* kg object going in the same direction with a speed of *32* m / s. If the collision is completely elastic, calculate the speed of the *20* kg object after collision.
   A. *27.405* m / s
   B. *24.665* m / s
   C. *19.184* m / s
   D. *21.924* m / s
   E. *16.443* m / s

7. An object of mass *5* kg going to the right with a speed of *30* m / s collides with a(n) *20* kg object going to the left with a speed of *11* m / s. If the collision is completely elastic, calculate the speed of the *20* kg object after collision.
   A. *7.56* m / s
   B. *7.02* m / s
   C. *5.94* m / s
   D. *5.4* m / s
   E. *4.32* m / s

8. An object of mass *5* kg going to the right with a speed of *28* m / s collides with a(n) *2* kg object at rest. After collision the *5* kg object moves with a speed of *20* m / s making an angle of *30* degree with the horizontal-right. Calculate the x-component of the velocity of the *2* kg object after collision.
   A. *26.699* m / s
   B. *34.708* m / s
   C. *24.029* m / s
   D. *16.019* m / s
   E. *32.038* m / s

9. An object of mass *3* kg going to the right with a speed of *29* m / s collides with a(n) *14* kg object at rest. After collision the *3* kg object moves with a speed of *15* m / s making an angle of *60* degree with the horizontal-right. Calculate the y-component of the velocity of the *14* kg object after collision.
   A. *-1.949* m / s
   B. *-2.505* m / s
   C. *-1.67* m / s
   D. *-3.34* m / s
   E. *-2.784* m / s

10. An object of mass *9* kg going to towards north with a speed of *27* m / s collides with a(n) *28* kg object going east with a speed of *39* m / s. After collision the *9* kg object moves with a speed of *20* m / s making an angle of *80* degree with the horizontal-right. Calculate the magnitude and direction of the velocity of the *28* kg object after collision.
    A. *34.161* m / s, *3.901* deg
    B. *37.956* m / s, *3.546* deg
    C. *34.161* m / s, *3.546* deg
    D. *45.548* m / s, *2.837* deg
    E. *37.956* m / s, *3.901* deg

7 Circular Motion and Law of Gravitation

Your goals for this chapter are to learn about the nature of circular motion and the relationships between force and motion for circular motion.

7.1 Circular Motion

Circular motion is motion in a circular path. A polar coordinate system is more appropriate than Cartesian coordinate system for this kind of motion. Because if Cartesian coordinate system is used, the value of the x-coordinate and y-coordinate of the position of the particle will change constantly. But if polar coordinate system is used, the r-coordinate which is distance between the center (origin) and the particle remains constant. Only the θ-coordinate which is the angle between the position vector of the particle and the positive x-axis changes. Thus, if polar coordinate system is used only the angle between the position vector of the particle and the positive x-axis has to be considered to describe the motion.

There are two units for the measurement of an angle: a degree and a radian. A *degree* is defined to be $(1/360)^{th}$ of a complete circle. That is, there are *360* degrees in one revolution. A degree is abbreviated as deg or as °. A *radian* is defined to be the degree measure of a central angle that subtends an arc-length equal to the radius of the circle. The radian is the SI unit of measurement for an angle. A radian is abbreviated as rad. Generally, if θ is a central angle in radians, r is the radius of the circle and s is the arc-length subtended by the central angle, then

$$\theta = s/r$$

Since a radian is a ratio between lengths, it is unit less. The number of radians in a complete circle may be obtained by dividing the circumference of a circle by the radius. There are 2π radians in a complete circle. A relationship between a degree and a radian can be obtained using the fact that one revolution is equal to *360* degrees or 2π radians.

$$\text{rad} = (180/\pi)\ \text{deg}$$

$$\text{deg} = (\pi/180)\ \text{rad}$$

Example: Convert

a) *120* degrees to radians

Solution:

$$60\ \text{deg} = 120 * \pi/180\ \text{rad} = 2\pi/3\ \text{rad}$$

---

b) π radians to degrees

Solution:

$$\pi\ \text{rad} = \pi * 180/\pi\ \text{deg} = 180\ \text{deg}$$

---

c) Convert 2 revolutions to degrees

Solution:

$$2\ \text{rev} = 2 * 360\ \text{deg} = 720\ \text{deg}$$

---

d) Convert 5 revolutions to radians

Solution:

$$5 \text{ rev} = 5 * 2\pi \text{ rad} = 10\pi \text{ rad}$$

Example: Calculate the radian measure of a central angle that subtends an arc-length of 10 cm in a circle of radius 20 cm.

Solution: $s = 10$ cm; $r = 20$ cm; $\theta = ?$

$$\theta = s/r = 10/20 = 0.5 \text{ rad}$$

7.2 Uniform Circular Motion

Uniform circular motion is motion in a circular path with a constant speed. The time taken for one complete revolution is called the *period* (abbreviated as T) of the motion. Its unit of measurement is second. The number of cycles executed per second is called the *frequency* (abbreviated as f) of the motion. Its unit of measurement is $1/\text{s}$ which is defined to be Hertz abbreviated as Hz. Frequency and period are inverses of each other.

$$f = 1/T$$

The number of radians executed per second is called the *angular speed* (abbreviated as ω) of the object. Its unit of measurement is rad / s. Since there are 2π radians in a cycle, angular speed is equal to 2π times frequency.

$$\omega = 2\pi f = 2\pi/T$$

The speed of the object (v) may be obtained as the ratio between the circumference of the circular path and the period of the motion.

$$v = 2\pi r/T = 2\pi r f = \omega r$$

Example: An object is revolving in a circular path of radius 4 m with a constant speed of 10 m / s.

a) How long does it take to make one complete revolution?

Solution: $r = 4$ m; $v = 10$ m / s; $T = ?$

$$v = 2\pi r/T$$

$$T = 2\pi r/v = 2\pi * 4/10 \text{ s} = 2.5 \text{ s}$$

---

b) How many cycles does it execute per second?

Solution: $f = ?$

$$f = 1/T = 1/2.5 \text{ Hz} = 0.4 \text{ Hz}$$

---

c) Calculate its angular speed.

Solution: $\omega = ?$

$$\omega = 2\pi f = 2\pi^* 0.4 \text{ rad/s} = 2.5 \text{ rad/s}$$

7.2.1 Acceleration of a Uniform Circular Motion

Uniform circular motion is an accelerated motion even though the speed is constant because direction is changing constantly. An acceleration caused by a change in direction only is called centripetal or radial acceleration. Direction of centripetal acceleration is always towards the center of the circular path. As the particle is displaced by a small arc-length $|\Delta s|$, the position vector and the velocity vectors of the particle rotate by the same small angle. The triangle formed by the length of the initial position vector, final position vector and the arc-length $|\Delta s|$; and the triangle formed by the length of the initial velocity, final velocity vector and the vector joining the tips of this vectors, which is change in velocity $\Delta\mathbf{v}$, are similar triangles. Thus corresponding sides of these two triangles are proportional.

$$|\Delta v|/v = |\Delta s|/r$$

$$|\Delta v| = v|\Delta s|/r$$

Dividing both sides by the time interval Δt during which this displacement took place

$$|\Delta v|/\Delta t = (v/r)(|\Delta s|/\Delta t)$$

But $|\Delta v|/\Delta t$ is equal to the magnitude of centripetal or radial acceleration a_c. And $|\Delta s|/\Delta t$ is equal to the speed of the object v.

$$a_c = v^2/r$$

The force responsible for centripetal acceleration is called centripetal force F_c. It is related with centripetal acceleration by Newton's second law.

$$F_c = ma_c = mv^2/r$$

Example: An object of mass 5 kg is revolving in a circular path of radius 6 m with a constant speed of 4 m/s.

a) Calculate its centripetal acceleration.

Solution: $m = 5$ kg; $r = 6$ m; $v = 4$ m/s; $a_c = ?$

$$a_c = v^2/r = 4^2/6 \text{ m/s}^2 = 2.7 \text{ m/s}^2$$

---

b) Calculate the centripetal force responsible for this acceleration.

Solution: $F_c = ?$

$$F_c = ma_c = 5 * 2.7 \text{ N} = 13.5 \text{ N}$$

---

c) Assuming the center of the circle is fixed at the origin of a coordinate system, determine the acceleration of the object when it is located at the intersection point of the circular trajectory and the positive x-axis.

Solution: West, because centripetal acceleration is always directed towards the center

---

7.3 Uniformly Accelerated Circular Motion

Uniformly accelerated circular motion is motion in a circular path where the speed of the object is changing with time at a constant rate.

7.3.1 Motion Variables of a circular motion

Angular position (θ) of a particle is angle formed between the position vector of the particle and the positive x-axis. Its unit of measurement is the radian.

Angular displacement ($\Delta\theta$) of a particle is defined to be change in the angular position of the particle. Its unit of measurement is radian.

$$\Delta\theta = \theta_f - \theta_i$$

Average angular velocity (ω_{av}) is defined to be change in angular displacement per a unit time. Its unit is rad / s.

$$\omega_{av} = (\theta_f - \theta_i) / \Delta t$$

Instantaneous angular velocity (ω) is angular velocity at a given instant of time.

Average angular acceleration (α_{av}) is change in angular velocity per a unit time. Its unit is rad / s^2.

$$\alpha_{av} = (\omega_f - \omega_i) / \Delta t$$

Instantaneous angular acceleration (α) is angular acceleration at a given instant of time.

7.3.2 Net Acceleration of a Uniformly Accelerated Circular Motion

A uniformly accelerated circular motion has two kinds of acceleration. One is acceleration due to change of direction which is called centripetal or radial acceleration (a_c). Its direction is towards the center. The other is acceleration due to change of speed and is called tangential acceleration (a_t). Its direction is tangent to the circle. Therefore centripetal acceleration and tangential acceleration are perpendicular to each other. The net acceleration is the vector sum of the centripetal and tangential accelerations. Since centripetal and tangential acceleration are perpendicular to each other, an expression for the magnitude of net acceleration (a_{net}) can be obtained by applying Pythagorean Theorem.

$$a_{net} = \sqrt{(a_c{}^2 + a_t{}^2)}$$

7.3.3 Relationship between Linear and Angular Variables

Distance or arc-length s, angular position θ and radius r are related by $s = r\theta$. Taking the changes of both sides of the equations, a relationship between linear displacement and angular displacement is obtained.

$$\Delta s = r\Delta\theta$$

Dividing both sides of this equation by the interval of time Δt during which these displacements took place, the equation $\Delta s/\Delta t = r\Delta\theta/\Delta t$ is obtained. But $\Delta s/\Delta t$ is equal to the linear speed v and $\Delta\theta/\Delta t$ is equal to the angular speed ω.

$$v = r\omega$$

Taking the change of both sides of this equation and then dividing by Δt the equation $\Delta v/\Delta t = r\Delta\omega/\Delta t$ is obtained. But $\Delta v/\Delta t$ is equal to the tangential acceleration a_t and $\Delta\omega/\Delta t$ is equal to the angular acceleration α.

$$a_t = r\alpha$$

7.3.4 Equations of a Uniformly Accelerated Circular Motion

Uniformly accelerated circular motion is motion with a constant angular acceleration. Equations of a uniformly accelerated motion are similar to the equations of a uniformly accelerated linear motion.

$$\omega_f = \omega_i + \alpha t$$

$$\Delta\theta = \omega_i\, t + \alpha t^2/2$$

$$\omega_f^2 = \omega_i^2 + 2\alpha\Delta\theta$$

$$\Delta\theta = (\omega_f + \omega_i)t/2$$

These equations involve five variables: initial angular velocity (ω_i), final angular velocity (ω_f), angular displacement ($\Delta\theta$), angular acceleration (α) and time (t). Only two of these equations are independent. Thus, if three of these variables are known, the other two can be calculated with the help of these equations.

Example: An object is revolving in a circular path of radius 2 m with a uniform angular acceleration. Its angular speed increased from 10 rad/s to 20 rad/s in 4 seconds.

a) Calculate its angular acceleration.

Solution: $\omega_i = 10$ rad/s; $\omega_f = 20$ rad/s; $t = 4$ s; $\alpha = ?$

$$\omega_f = \omega_i + \alpha t$$

$$\alpha = (\omega_f - \omega_i)/t = (20 - 10)/4 \text{ rad/s}^2 = 2.5 \text{ rad/s}^2$$

---

b) Calculate its tangential acceleration.

Solution: $r = 2$ m; $a_t = ?$

$$a_t = r\alpha = 2 * 2.5 \text{ m/s}^2 = 5 \text{ m/s}^2$$

---

c) Calculate its angular displacement.

Solution: $\Delta\theta = ?$

$$\Delta\theta = \omega_i t + \alpha t^2/2 = (10 * 4 + 2.5 * 4^2/2) \text{ rad} = 60 \text{ rad}$$

---

d) Calculate the distance travelled.

Solution: $\Delta s = ?$

$$\Delta s = r\Delta\theta = 2 * 60 \text{ m} = 120 \text{ m}$$

---

e) Calculate its initial speed.

Solution: $v_i = ?$

$$v_i = r\omega_i = 2 * 10 \text{ m/s} = 20 \text{ m/s}^2$$

---

f) Calculate its initial centripetal acceleration.

Solution: $a_c = ?$

$$a_c = v_i^2/r = 20^2/2 \text{ m/s}^2 = 200 \text{ m/s}^2$$

---

g) Calculate the magnitude of its net acceleration.

Solution: $a_{net} = ?$

$$a_{net} = \sqrt{(a_c^2 + a_t^2)} = \sqrt{(200^2 + 5^2)} \text{ m/s}^2 = 200.1 \text{ m/s}^2$$

7.4 Practice Quiz 7.1

Choose the best answer. Answers can be found at the back of the book.

1. Convert *2.2* radians to degrees.
   A. *100.841* deg
   B. *163.866* deg
   C. *113.446* deg
   D. *75.63* deg
   E. *126.051* deg

2. Convert *0.8* revolution to radian.
   A. *5.027* rad
   B. *4.524* rad
   C. *6.535* rad
   D. *6.032* rad
   E. *8.042* rad

3. Calculate the radian measure of a central angle that subtends (opens) an arc-length of 9 m in a circle of radius 1.5 m.
   A. 5.4 rad
   B. 7.8 rad
   C. 4.8 rad
   D. 6 rad
   E. 8.4 rad

4. An object is revolving in a circular path of radius 18 m with a uniform speed of 13 m / s. How long does it take to make one complete revolution?
   A. 7.83 s
   B. 5.22 s
   C. 8.7 s
   D. 6.96 s
   E. 6.09 s

5. An object is revolving in a circular path of radius 6 m with a uniform speed. If it takes 1.25 s to make one complete revolution, calculate the speed with which it is revolving.
   A. 18.096 m / s
   B. 27.143 m / s
   C. 24.127 m / s
   D. 30.159 m / s
   E. 21.112 m / s

6. An object is revolving in a circular path of radius 4 m with a uniform speed. If it makes 50 revolutions in 6 seconds, calculate its period.
   A. 0.144 s
   B. 0.132 s
   C. 0.168 s
   D. 0.12 s
   E. 0.084 s

7. An object is revolving in a circular path of radius 14 m with a uniform speed. If it makes 17 cycles per second, calculate its centripetal acceleration.
   A. 143756.71 m / s^2
   B. 223621.549 m / s^2
   C. 191675.613 m / s^2
   D. 159729.678 m / s^2
   E. 111810.774 m / s^2

8. An object of mass *0.6* kg is revolving in a circular path of radius *16* m with a uniform speed of *13* m / s. Calculate the force responsible for this motion.
   A. *8.239* N
   B. *6.337* N
   C. *3.802* N
   D. *6.971* N
   E. *5.07* N

9. A uniformly accelerated circular motion
   A. is motion in a circular path where the angular speed is changing.
   B. is motion in a circular path where the angular speed changes at constant rate with time.
   C. is motion in a circular path where the speed is changing.
   D. is motion in a circular motion where the angular acceleration changes at a constant rate with time.
   E. is motion in a circular path with constant centripetal acceleration.

10. Which of the following statements is not a correct statement.
    A. Uniformly accelerated circular motion is motion with constant angular acceleration.
    B. Tangential and centripetal acceleration of a circular motion are always perpendicular to each other.
    C. Uniform circular motion is not an accelerated motion.
    D. The direction of tangential acceleration of a circular motion is always tangent to the circular path.
    E. Uniform circular motion is motion with a constant speed.

11. A wheel of radius *18* m was rolled for a distance of *8* m. Calculate the angular displacement of a point on the rim of the wheel.
    A. *0.4* rad
    B. *0.622* rad
    C. *0.311* rad
    D. *0.533* rad
    E. *0.444* rad

12. The angular speed of an object revolving in a circular path of radius *2.7* m changed from *6* rad/s to *3* rad/s uniformly in *0.5* seconds. Calculate the distance travelled.
    - A. *6.075* m
    - B. *5.468* m
    - C. *3.645* m
    - D. *7.898* m
    - E. *7.29* m

13. Starting with an angular speed of *18* rad/s, an object was accelerated uniformly on a circular path of radius *0.4* m with an acceleration of *8* rad/s^2 for *21* s. Calculate its final angular speed.
    - A. *111.6* rad/s
    - B. *204.6* rad/s
    - C. *241.8* rad/s
    - D. *186* rad/s
    - E. *223.2* rad/s

14. An object is revolving in a circular path of radius *3.6* m with an angular acceleration of *24* rad / s ^{2}. At a time when its angular speed is *30* rad / s, calculate the net acceleration of the object.
    A. *4213.497* m / s ^{2}
    B. *3565.267* m / s ^{2}
    C. *3889.382* m / s ^{2}
    D. *4537.613* m / s ^{2}
    E. *3241.152* m / s ^{2}

7.5 Law of Gravitation

Newton's Law of Gravitation states that any two objects in the universe attract each other with a gravitational force that is proportional to the product of their masses and inversely proportional to the square of the distance separating their centers. The direction of the force is along the line joining their centers. If two objects of masses m_1 and m_2 are separated by a distance r, then

$$F_g = Gm_1m_2/r^2$$

F_g is the gravitational force exerted by one on the other. G is a universal constant called universal gravitational constant. Its value is $6.67 * 10^{-11}$ N m ^{2}/ kg ^{2}.

$$G = 6.67 * 10^{-11} \text{ N m}^2/\text{kg}^2$$

The force exerted by m_1 on m_2 and by m_2 on m_1 are action reaction forces. They have equal magnitude but opposite direction.

Example: A $3 * 10^5$ kg object is located *300* m to the left of a $2 * 10^5$ kg object.

a) Calculate the magnitude and direction of the gravitational force exerted by the $3 * 10^5$ kg object on the $2 * 10^5$ kg object.

   Solution: The direction of the force is west because it is an attractive force.

   $m_1 = 3 * 10^5$ kg; $m_2 = 2 * 10^5$ kg; $r = 300$ m; $F_g = ?$

   $$F_g = Gm_1m_2/r^2 = 6.67 * 10^{-11} * 3 * 10^5 * 2 * 10^5/300^2 \text{ N} = 0.000044 \text{ N}$$

---

b) Calculate the magnitude and direction of the force exerted by the $2 * 10^5$ kg object on the $3 * 10^5$ kg object.

   Solution: The two forces are action reaction forces. Therefore the direction of the force is east and the magnitude of the force is *0.000044* N.

Example: Consider the motion of earth around the sun. The masses of earth and sun are $5.98 * 10^{24}$ kg and $1.991 * 10^{30}$ kg respectively. The distance between earth and sun is $1.496 * 1o^{11}$ m.

a) What is the kind of force that keeps earth rotating around earth?

Solution: Gravitational force exerted by sun on earth.

---

b) Calculate the magnitude of the gravitational force exerted by sun on earth.

Solution: $m_e = 5.98 * 10^{24}$ kg; $m_s = 1.991 * 10^{30}$ kg; $r = 1.496 * 10^{11}$ m; $F_g = ?$

$F_g = Gm_e m_s / r^2 = 6.67 * 10^{-11} * 5.98 * 10^{24} * 1.991 * 10^{30} / (1.496 * 10^{11})^2$ N $= 35.5 * 10^{21}$ N

---

c) Calculate the centripetal acceleration of earth.

Solution: The centripetal force is supplied by gravitational force.

$a_c = ?$

$$F_c = F_g = m_e a_c$$

$$a_c = F_g / m_e = 35.5 * 10^{21} / (5.98 * 10^{24}) \text{ m/s}^2 = 5.9 * 10^{-3} \text{ m/s}^2$$

---

d) Calculate the speed with which earth is rotating around sun.

Solution: $v = ?$

$$a_c = v^2 / r$$

$$v = \sqrt{(ra_c)} = \sqrt{(1.496 * 10^{11} * 5.9 * 10^{-3})} \text{ m/s} = 2.97 * 10^4 \text{ m/s}$$

---

e) Calculate the time earth takes to make one complete revolution around the sun.

Solution: $T = ?$

$$T = 2\pi r / v = 2\pi * 1.496 * 10^{11} / (2.97 * 10^4) \text{ s} = 3.16 * 10^7 \text{ s}$$

$$= 3.16 * 10^7 / 60 / 60 / 24 \text{ days} = 365.7 \text{ days}$$

7.6 Problems Involving Circular Motion

The following are examples involving circular motion.

Example: A car is revolving in a circular path of radius *20* m. The coefficient of friction between the ground and the tires of the car is *0.2*. Calculate the maximum speed with which the car can make it without sliding.

Solution: The centripetal force that keeps the car revolving in the circular path is friction. If the car acquires a speed which requires a centripetal force bigger than the force of friction, it will slide. The normal force exerted by the ground is equal to the weight of the object because the normal force has to balance the weight: $N = m|g|$

$r = 20$ m; $\mu = 0.2$; $v_{max} = ?$

$$F_c = m{v_{max}}^2 / r = f = \mu N = \mu m|g|$$

$$v_{max} = \sqrt{(\mu r|g|)} = \sqrt{(0.2 * 20 * 9.8)}\ \mathrm{m/s} = 6.3\ \mathrm{m/s}$$

Example: An object of mass *0.04* kg revolves on a circular path of radius *0.2* m on a friction-less table by means of a string. The string passes through a hole at the center of the circular path and then connected to a hanging object of mass *2* kg. Calculate the speed with which the *0.04* kg object is revolving.

Solution: The Centripetal force that keeps the object is supplied by the tension (T) in the string. The tension in the string is equal to the weight of the hanging object.

$m = 0.04$ kg; $M = 2$ kg; $r = 0.2$ m; $v = ?$

$$F_c = mv^2/r = T = M|g|$$

$$v = \sqrt{(M|g|r/m)} = \sqrt{(2 * 9.8 * 0.2/0.04)}\ \mathrm{m/s} = 9.9\ \mathrm{m/s}$$

Example: A pendulum of length *2* m is revolving in a horizontal circle. The string makes an angle of *30°* with the vertical. Calculate the speed with which the object is revolving around the circle.

Solution: The centripetal force is supplied by the horizontal component of the tension (T) in the string. The vertical component of the tension supports the weight of the pendulum. The radius of the circle may be obtained as the horizontal projection of the length of the string.

$l = 2$ m; $\theta = 90° - 30° = 60°$ (angle with respect to the horizontal); $v = ?$

$$F_c = T_x = T \cos\theta = mv^2/r$$

$$v^2 = Tr \cos\theta/m$$

$$r = l \cos\theta$$

Substituting for r

$$v^2 = Tl \cos^2\theta/m$$

$$T_y = T \sin\theta = m|g|$$

$$T = m|g|/\sin\theta$$

Substituting for T

$$v^2 = lm|g| \cos^2\theta/(m \sin\theta) = l|g| \cos^2\theta/\sin\theta = 2 * 9.8 * \cos^2 60°/\sin 60°\ \mathrm{m^2/s^2} = 5.7\ \mathrm{m^2/s^2}$$

$$v = \sqrt{(5.7)}\ \mathrm{m/s} = 2.4\ \mathrm{m/s}$$

Example: An object of mass *5* kg is revolving in a vertical circle of radius *2* m by means of a string.

a) Calculate the minimum speed at the top of the circle with which the object can make it on the circular path without falling.

Solution: The centripetal force for this motion is supplied by the tension in the string and the weight of the object. The minimum speed is the speed at which the string just slacks down. Therefore for the minimum speed at the top of the circle only the weight of the object contributes to the centripetal force.

Solution: $m = 5$ kg; $r = 2$ m; $v_{min} = ?$

$$F_c = m|g| = mv_{min}^2/r$$

$$v_{min} = \sqrt{(r|g|)} = \sqrt{(2 * 9.8)}\ \mathrm{m/s} = 4.4\ \mathrm{m/s}$$

---

b) Assuming the speed at the top is the minimum speed, calculate its speed at the bottom of the circle.

Solution: The forces acting on the object are tension in the string and gravity. The tension in the string does not contribute to the work done because the tension is perpendicular to the trajectory. Since gravity is the only force that contributes to the work done, principle of conservation of mechanical energy can be used. Let the origin of the coordinate system be fixed at the bottom of the circle.

$y_f = 0$; $y_i = 2r = 4$ m; $v_i = v_{min} = 4.4$ m/s; $v_f = v_{bot} = ?$

$$mv_i^2/2 + m|g|y_i = mv_f^2/2 + m|g|y_f$$

$$4.4^2/2 + 9.8 * 4 = v_{bot}^2/2$$

$$v_{bot} = 9.9\ \mathrm{m/s}$$

7.7 Practice Quiz 7.2

Choose the best answer. Answers can be found at the back of the book.

1. Gravitational force between two objects is
   A. an attractive force which is proportional to the product of the masses of the objects and inversely proportional to the square of the distance separating the two objects.
   B. an attractive force which is proportional to the sum of the masses of the objects and inversely proportional to the square of the distance separating the two objects.
   C. an attractive force which is proportional to the sum of the masses of the objects and inversely proportional to the distance separating the two objects.
   D. a repulsive force which is proportional to the product of the masses of the objects and inversely proportional to the distance separating the two objects.
   E. an attractive force which is proportional to the product of the masses of the objects and inversely proportional to the distance separating the two objects.

2. By how much would the gravitational force between two objects be multiplied if the mass of one of the objects is multiplied by a factor of *4* and the mass of the other object is multiplied by a factor of *9?*
   A. *13*
   B. *36*
   C. *0.444*
   D. *0.028*
   E. *2.25*

3. Object *1* and Object *2* are located on the x-axis at $x = 300$ m and $x = 800$ m respectively. What is the direction of the gravitational force exerted by object *1* on object *2*?
   A. South
   B. It cannot be determined
   C. East
   D. North
   E. West

4. Calculate the distance between an object of mass *9000* kg and an object of mass *9000* kg, if they are exerting a gravitational force of *2e-9* N on each other.
   A. *1479.221* m
   B. *2136.652* m
   C. *2301.01* m
   D. *1643.578* m
   E. *1972.294* m

5. A satellite of mass *800000* kg is revolving around earth at an altitude of *3000* km above the surface of earth. Earth has a mass of *5.98e24* kg and a radius of *6.38e6* m. Calculate the speed with which the satellite is revolving around earth.
   A. *8477.259* m / s
   B. *9129.356* m / s
   C. *6520.968* m / s
   D. *5216.775* m / s
   E. *5868.871* m / s

6. A car of mass *3000* kg is turning on a curve of radius of curvature *80* m. The coefficient of friction between the tires of the car and the ground is *0.25*. Calculate the maximum speed by which the car can make it without sliding.
   A. *11.2* m / s
   B. *14* m / s

C. *9.8* m / s
D. *19.6* m / s
E. *12.6* m / s

7. An object of mass *0.1* kg is revolving in a circular path of radius *0.2* m on a friction less table by means of a string which is attached to a hanging object of mass *1* kg (the string is attached to the hanging object through a hole on the table at the center of the circular path). Calculate the speed with which the object on the table is revolving on the circular path.
   A. *3.984* m / s
   B. *5.755* m / s
   C. *4.427* m / s
   D. *6.198* m / s
   E. *5.313* m / s

8. A *0.8* kg pendulum of length *0.4* m that makes an angle of *10* deg with the vertical is revolving in a horizontal circle. Calculate the speed with which it is revolving on the horizontal circle.
   A. *0.346* m / s
   B. *0.45* m / s
   C. *0.381* m / s
   D. *0.277* m / s
   E. *0.485* m / s

9. An object of mass *4.5* kg is revolving in a vertical circle of radius *0.5* m. Calculate the minimum speed at the top of the circle by which the object can make it without the string slacking.
   A. *2.878* m / s
   B. *1.992* m / s
   C. *2.656* m / s
   D. *2.214* m / s
   E. *3.099* m / s

10. An object of mass *1.5* kg is revolving in a vertical circle of radius *0.9* m. If its speed at the top of the circle is the minimum speed by which it can make it without the string slacking, calculate its speed at the bottom of the circle.
   A. *5.313* m / s
   B. *8.633* m / s
   C. *9.297* m / s
   D. *6.641* m / s
   E. *7.305* m / s

8 Rotational Equilibrium and Rotational Dynamics

Your goals for this chapter are to learn about torque and the relationships between torque and rotational motion.

8.1 Torque

Torque is a vector physical quantity used as a measure of the rotational effect of force. It is proportional to the magnitude of the force and to the perpendicular distance between the point of rotation and the line of action of the force.

$$|\tau| = |F|r_{\perp}$$

Where $|\tau|$ and $|F|$ represent the magnitude of the torque and force respectively; and $r_\perp$ represents the perpendicular distance between the point of rotation and the line of action of the force. If r is the distance between the point of rotation and the point of application of the force and θ is the angle between the position vector of the point of application of force with respect to the point of rotation (The vector whose tail is at the point of rotation and whose head is at the point of application of force), then $r_\perp = r \sin \theta$. Thus

$$|\tau| = |F| r \sin \theta$$

The direction of torque is perpendicular to the plane determined by the force vector and the position vector of the point of application of force (with respect to the point of rotation). It is perpendicularly out if the tendency of the force is to produce counterclockwise rotation and perpendicularly in if the tendency of the force is to produce clockwise rotation. The component of torque (τ) is taken to be positive if the tendency of the force is to produce counterclockwise rotation and negative if the tendency of the force is to produce clockwise rotation.

$$\tau = \pm |F| r \sin \theta$$

The unit of measurement for torque is Newton meter (N m).

Example: A horizontal lever of length 4 m is pivoted at its center. A vertically downward force of 5 N is applied at a distance of 0.5 m to the right of the pivot. Calculate the torque acting on the lever.

Solution: The force has the tendency of producing clockwise rotation. Therefore the torque is negative.

$|F| = 5$ N; $r = 0.5$ m; $\theta = 90°$; $\tau = ?$

$$\tau = -|F| r \sin \theta = -5 * 0.5 * \sin 90° \text{ N m} = -2.5 \text{ N m}$$

Example: A horizontal lever of length 6 m is pivoted at its center. A force of 10 N with an upward vertical component makes an angle of $30°$ with the positive x-axis is acting at the right end of the lever. Calculate the torque acting on it.

Solution: The torque is positive because the tendency of the force is to produce counterclockwise rotation.

$|F| = 10$ N; $r = 3$ m; $\theta = 30°$; $\tau = ?$

$$\tau = |F| r \sin \theta = 10 * 3 * \sin 30° \text{ N m} = 15 \text{ N m}$$

Example: A horizontal lever of length *3* m is pivoted at its center. A horizontal force of *40* N is pulling to the left at its left end. Calculate the torque acting on it.

Solution: This force does not have any rotational effect on the lever. The torque is expected to be zero.

$|F| = 40$ N; $r = 1.5$ m; $\theta = 0°$; $\tau = ?$

$$\tau = |F| r \sin \theta = 40 * 1.5 * \sin 0° \text{ N m} = 0$$

8.1.1 Net Torque

Net torque acting on an object is the vector sum of all the torques acting on the object.

$$\boldsymbol{\tau}_{net} = \boldsymbol{\tau}_1 + \boldsymbol{\tau}_2 + \ldots$$

If the object is rotating in a plane, all the torques have the same line of action (either perpendicularly out or perpendicularly in) and this vector equation can be described by a single component equation.

$$\tau_{net} = \tau_1 + \tau_2 + \ldots$$

Example: A horizontal uniform lever of length *4* m is pivoted at its center. A force of *5* N is pulling vertically down ward at the right end of the lever. A force of *10* N that makes an angle of *37°* with the negative x-axis is pulling the lever upward at its right end. A force of *2* N is pushing vertically downward at a distance of *1* m to the left of the pivot. Calculate the net torque acting on the lever and determine whether it is rotating clockwise or counterclockwise.

Solution: The torque due to the *5* N force (τ_1) is negative because it has a tendency of causing clockwise rotation. The torque due to the *10* N force (τ_2) is positive because its tendency is to cause counterclockwise rotation. The torque due to the *2* N force (τ_3) is positive because it has a tendency of causing counterclockwise rotation.

$|F_1| = 5$ N; $\theta_1 = 90°$; $r_1 = 2$ m; $|F_2| = 10$ N; $\theta_2 = 37°$ (technically $(180 - 37)°$ but the sine of supplementary angles are equal); $r_2 = 2$ m; $|F_3| = 2$ N; $\theta_3 = 90°$; $r_3 = 1$ m; $\tau_{net} = \tau_1 + \tau_2 + \tau_3 = ?$

$$\tau_1 = |F_1| r_1 \sin \theta_1 = -5 * 2 * \sin 90° \text{ N m} = -10 \text{ N m}$$

$$\tau_2 = |F_2| r_2 \sin \theta_2 = 10 * 2 * \sin 37° \text{ N m} = 12 \text{ N m}$$

$$\tau_3 = |F_3| r_3 \sin \theta_3 = 2 * 1 * \sin 90° \text{ N m} = 2 \text{ N m}$$

$$\tau_{net} = \tau_1 + \tau_2 + \tau_3 = (-10 + 12 + 2) \text{ N m}$$

It is rotating counterclockwise because the net torque is positive.

8.2 Rotational Equilibrium

There are two kinds of equilibriums: translational and rotational equilibrium. An object is said to be in *translational equilibrium* if it is either at rest or moving in a straight line with a constant speed. The *condition of translational equilibrium* states that an object will be in translational equilibrium if the net force acting on it is zero.

$$\sum \mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \ldots = \mathbf{0}$$

If a vector is equal to zero, then its components are also equal to zero. The following equations are statements of the condition of translational equilibrium in component form.

$$\sum F_x = F_{1x} + F_{2x} + \ldots = 0$$

$$\sum F_y = F_{1y} + F_{2y} + \ldots = 0$$

An object is said to be in *rotational* **equilibrium** if it is either at rest or rotating with a constant angular speed. The *condition of rotational equilibrium* states that an object will be in rotational equilibrium if the net torque acting on it is equal to zero.

$$\sum \boldsymbol{\tau} = \boldsymbol{\tau}_1 + \boldsymbol{\tau}_2 + \ldots = \mathbf{0}$$

If the rotation is in a plane, the torques will have the same line of action (perpendicularly out or in) and this vector equation can be represented by a single (one dimensional) component equation.

$$\sum \tau = \tau_1 + \tau_2 + \ldots = 0$$

If an object is in equilibrium both translational and rotational conditions of equilibrium apply. For an object in rotational equilibrium, an arbitrary point may be chosen as a point of rotation.

Example: A uniform lever of length *5* m pivoted at its center is in equilibrium. An upward force of *4* N that makes an angle of *53°* with the positive x-axis is acting at its right end. A vertically downward force of *20* N is acting at a point *2* m to the right of its pivot. An unknown vertically downward force is acting at a point a distance of *1.5* m to the left of the pivot. Assume the weight of the lever is negligible

a) Calculate the magnitude of the unknown force.

Solution: The forces acting on the lever are the force exerted by the pivot and the listed forces. Thus there are two unknown forces (including the force due to the pivot). If the point of rotation is chosen at the point of rotation, the force due to the pivot will not contribute to the torques acting on the object. Thus, the application of the condition of rotational equilibrium with the pivot as point of rotation yields an equation with one unknown. Let the magnitude of the unknown force be denoted by $|F|$ and the force due to the pivot be denoted by $\mathbf{F}_p$

$$|F_1| = 4 \text{ N};\ \theta_1 = 53°;\ r_1 = 2.5 \text{ m};\ |\mathrm{F}_2| = 20 \text{ N};\ \theta_2 = 90°;\ r_2 = 2 \text{ m};\ \theta_F = 90°;\ r_F = 1.5;\ |F| = ?$$

$$\tau_1 + \tau_2 + \tau_F = 0$$

$$|F_1| r_1 \sin\theta_1 - |F_2| r_2 \sin\theta_2 + |F| r_F \sin\theta_F = 0$$

$$|F| r_F \sin\theta_F = -|F_1| r_1 \sin\theta_1 + |F_2| r_2 \sin\theta_2 = (-4 * 2.5 * \sin 53° + 20 * 2 * \sin 90°) \text{ N m} = 32 \text{ N m}$$

$$|F|(1.5 \text{ m}) * \sin 90° = 32 \text{ N m}$$

$$|F| = 32 / 1.5 \text{ N} = 21.3 \text{ N}$$

---

b) Calculate the x and y components of the force exerted by the pivot.

Solution: The x and y components of the force exerted by the pivot can be obtained by applying the condition of translational equilibrium in component form.

$F_{1x} = 4 * \cos 53°$ N $= 2.4$ N; $F_{1y} = 4 * \sin 53°$ N $= 3.2$ N; $F_{2x} = 20 * \cos -90$ N $= 0$; $F_{2y} = 20 * \sin -90$ N $= -20$ N; $F_x = 23.2 * \cos -90 = 0$; $F_y = 23.2 * \sin -90$ N $= -23.2$ N; $F_{px} = ?$; $F_{py} = ?$

$$F_{px} + F_{1x} + F_{2x} + F_x = 0$$

$$F_{px} + (2.4\ N) + (0) + (0) = 0$$

$$F_{px} = -2.4 \text{ N}$$

$$F_{py} + F_{1y} + F_{2y} + F_y = 0$$

$$F_{px} + (3.2\ N) + (-20) + (-23.2) = 0$$

$$F_{py} = 40 \text{ N}$$

8.2.1 Center of Gravity

The *center of gravity* of an object is the point at which the object can be balanced. For uniform (constant density) objects, the center of gravity is the same as the geometrical center. For example the center of gravity of a uniform circular disc is at the center of the circle. For problems involving torque, where the point of application of the force has to be specified, the weight of an object can be assumed to act at the center of gravity of the object.

The fact that an object can be balanced at the center of gravity indicates that the torque due to the balancing force (which is equal to the weight) is equal to the sum of the torques of the weights of the particles comprising the object. Taking the torques about the origin of a certain coordinate system, the perpendicular distances are equal to the x-coordinates because the forces are vertical.

$$M|g|x_g = m_1|g|x_1 + m_2|g|x_2 + \ldots$$

Where $M = m_1 + m_2 + \ldots$, is the total mass of the object, x_i is the x-coordinate of the i^{th} particle and m_i is the mass of the i^{th} particle. Thus, the x-coordinate of its center of gravity is given by

$$x_g = (\Sigma_i\, m_i x_i) / (\Sigma_i\, m_i) = (m_1 x_1 + m_2 x_2 + \ldots) / (m_1 + m_2 + \ldots)$$

A similar expression can be obtained for the y-coordinate of the center of gravity by rotating the object along with the coordinate system by $90°$.

$$y_g = (\Sigma_i\, m_i y_i) / (\Sigma_i\, m_i) = (m_1 y_1 + m_2 y_2 + \ldots) / (m_1 + m_2 + \ldots)$$

Example: A system is comprised of *4* particles. The first one has a mass of *2* kg and is located at *(-4, 6)* m. The second one has a mass of *4* kg and is located at *(0, -5)* m. The third one has a mass of *3* kg and is located at *(2, 3)* m. The fourth one has a mass of *1* kg and is located at *(-2, -2)* m. Calculate the x- and y-coordinates of the center of gravity of the system.

Solution: $m_1 = 2$ kg; $(x_1, y_1) = (-4, 6)$ m; $m_2 = 4$ kg; $(x_2, y_2) = (0, -5)$ m; $m_3 = 3$ kg; $(x_3, y_3) = (2, 3)$ m; $m_4 = 1$ kg; $(x_4; y_4) = (-2, -2)$ m

$$x_g = (m_1x_1 + m_2x_2 + \ldots)/(m_1 + m_2 + \ldots)$$

$$= (2 * -4 + 4 * 0 + 3 * 2 + 1 * -2)/(2 + 4 + 3 + 1) \text{ m} = -0.4 \text{ m}$$

$$y_g = (m_1y_1 + m_2y_2 + \ldots)/(m_1 + m_2 + \ldots)$$

$$= (2 * 6 + 4 * -5 + 3 * 3 + 1 * -2)/(2 + 4 + 3 + 1) \text{ m} = -0.1 \text{ m}$$

8.3 Practice Quiz 8.1

Choose the best answer. Answers can be found at the back of the book.

1. Torque is a physical quantity used as a measure of
   A. a force's ability to displace an object
   B. a force's rotational effect.
   C. a force's ability to change the angular acceleration of an object.
   D. a force's ability to deform an object
   E. the amount of motion an object has.

2. Which of the following is a correct statement?
   A. Moment of inertia of an object may have different values for different axis of rotations.
   B. Moment of inertia of an object does not depend on the mass of the object.
   C. Moment of inertia of a particle is equal to the product of its mass and its perpendicular distance from the axis of rotation.
   D. Moment of inertia of an object has the same value for different axis of rotations.
   E. Moment of inertia of a particle decreases as the perpendicular distance between the axis of rotation and the particle increases.

3. A horizontal lever of length *2.2* m is pivoted at its mid-point. A downward force of *3* N is acting at the right end of the lever. Calculate the torque acting on the lever.
   A. *2.97* N m
   B. *-3.3* N m
   C. *3.96* N m
   D. *4.62* N m
   E. *-4.29* N m

4. A horizontal lever of length *6* m is pivoted at its mid-point. A *40* N upward force (force with upward vertical component) that makes an angle of *50* deg with the horizontal-left is acting at a point *0.3* m away from the left end of lever. Calculate the torque acting on the lever.
   A. *-99.279* N m
   B. *82.733* N m
   C. *99.279* N m
   D. *-82.733* N m
   E. *-49.64* N m

5. A uniform horizontal lever of length *6.0* m is pivoted at its mid-point. The following forces are acting on the lever: a *9* N vertically upward force acting at the right end of the lever, a *5* N vertically downward force acting at the left end of the lever, and a *6* N vertically downward force acting at a point on the lever *0.6* m away from the right end. Calculate the net torque acting on the lever.
   A. *27.6* N m
   B. *24.84* N m
   C. *16.56* N m
   D. *19.32* N m
   E. *22.08* N m

6. A uniform horizontal lever of length *3.6* m is pivoted at its mid-point. The following forces are acting on the lever: a *2* N vertically downward force acting at the right end of the lever, a *11* N downward force (with downward vertical component) that makes an angle of *20* deg with the horizontal-right acting at the left end of the lever, and a *3* N vertically upward force acting at a point on the lever *0.6* m away from the left end. Calculate the net torque acting on the lever.
   A. *-0.428* N m
   B. *-0.385* N m
   C. *-0.342* N m
   D. *-0.3* N m
   E. *-0.599* N m

7. A uniform horizontal lever of length *4.8* m is pivoted at its mid-point. A vertically downward force of *6* N is acting at the right end of the lever. An unknown vertically downward force is acting at a distance of *0.6* m to the left of the pivot. If the lever is in equilibrium, calculate the force exerted by the pivot (fulcrum) on the lever.
   A. *24* N
   B. *30* N
   C. *27* N
   D. *18* N
   E. *39* N

8. A uniform horizontal lever of length *2.4* m pivoted at its mid-point is in equilibrium. The following forces are acting on the lever: a *5* N downward force (force with downward vertical component) that makes an angle of *30* deg with the horizontal-left acting at the right end of the lever, an unknown vertically downward force acting at the left end of the lever, and a *10* N vertically downward force acting at a point on the lever *0.7* m away from the right end. Calculate the magnitude of the unknown force.
   A. *4.667* N
   B. *6.667* N
   C. *9.333* N
   D. *7.333* N
   E. *8* N

9. Three particles of masses *4* kg, *7* kg and *8* kg are located at the points (*0, 1*) m, (*16, -4*) m, and (*2, -3*) m respectively. Determine the location of the center of gravity of the three particles.
   A. (*7.411, -2.274*) m
   B. (*7.411, -2.526*) m
   C. (*6.737, -2.274*) m
   D. (*8.084, -2.021*) m
   E. (*6.737, -2.526*) m

10. The center of gravity of two particles of masses *18* kg and *11* kg is located at the point (*4, -1*) m. The *18* kg particle is located at the point (*4, 3*) m. Find the location of the *11* kg particle.
    A. (*3.2, -7.545*) m
    B. (*2.8, -4.527*) m
    C. (*4, -7.545*) m
    D. (*4, -8.3*) m
    E. (*3.2, -8.3*) m

8.4 Rotational Dynamics

Rotational dynamics is the study of accelerated rotational motion.

8.4.1 Relationship between torque and Angular Acceleration

Let's consider an object of mass m rotating in a circular path of radius $r_\perp$ with an angular acceleration of α under the influence of a tangential force F_t. The torque acting on the object about the center of the circle is equal to the product of the tangential force and the perpendicular distance between the force and the center of the circle: $\tau = F_t r_\perp$. And from Newton's second law: $F_t = ma_t$. But $a_t = r_\perp \alpha$. Therefore

$$\tau = m r_\perp{}^2 \alpha$$

The product $mr_{\perp}^2$ is called the moment of inertial of the object about the given axis and denoted by I.

$$I = mr_{\perp}^2$$

The unit of measurement for moment of inertia is kg m^2. The relationship between torque and angular acceleration is that torque is proportional to angular acceleration with the constant of proportionality being the moment of inertia of the object about the given axis. Moment of inertia of an object is a constant for a given axis of rotation but different for different axes of rotation.

$$\tau = I\alpha$$

Example: An object of mass 5 kg is revolving in a circular path of radius 2 m with an angular acceleration of 10 rad/s^2.

a) Calculate its moment of inertia about an axis passing through the center of the circle perpendicularly.

Solution: $m = 5$ kg; $r_{\perp} = 2$ m; $I = ?$

$$I = mr_{\perp}^2 = 5 * 2 \text{ kg m}^2 = 20 \text{ kg m}^2$$

---

b) Calculate the torque acting on the object about an axis that passes through the center of the circle perpendicularly.

Solution: $\alpha = 10\ \mathrm{rad/s^2}$; $\tau = ?$

$$\tau = I\alpha = 20 * 10\ \mathrm{N\ m} = 200\ \mathrm{N\ m}$$

For a system involving more than one particle, the moment of inertia of the system is equal to the sum of the moment of inertias of the individual particles.

$$I = \sum_i m_i r_{i\perp}^2 = m_1 r_{1\perp}^2 + m_2 r_{2\perp}^2 + \ldots$$

Example: A system consists of three particles. The first has a mass of 6 kg and is located at *(2, 3)* m. The second has a mass of 4 kg and is located at *(-2, 4)* m. The third has a mass of 5 kg and is located at *(0, -3)* m.

a) Calculate the moment of inertia of the system about the x-axis.

Solution: If the x-axis is the axis of rotation, then the perpendicular distance between the axis and a particle is equal to the absolute value of the y-coordinate of the particle.

$$m_1 = 6\ \mathrm{kg};\ r_{1\perp} = 3\ \mathrm{m};\ m_2 = 4\ \mathrm{kg};\ r_{2\perp} = 4\ \mathrm{m};\ m_3 = 5\ \mathrm{kg};\ r_{3\perp} = 3\ \mathrm{m};\ I_x = ?$$

$$I_x = m_1 r_{1\perp}^2 + m_2 r_{2\perp}^2 + m_3 r_{3\perp}^2 = (6 * 3^2 + 4 * 4^2 + 5 * 3^2)\ \mathrm{kg\ m^2} = 163\ \mathrm{kg\ m^2}$$

---

b) Calculate the moment of inertia of the system about the y-axis.

Solution: If the y-axis is the axis of rotation, then the perpendicular distance between the axis and a particle is equal to the absolute value of the x-coordinate of the particle.

$$m_1 = 6\ \mathrm{kg};\ r_{1\perp} = 2\ \mathrm{m};\ m_2 = 4\ \mathrm{kg};\ r_{2\perp} = 2\ \mathrm{m};\ m_3 = 5\ \mathrm{kg};\ r_{3\perp} = 0\ \mathrm{m};\ I_x = ?$$

$$I_y = m_1 r_{1\perp}^2 + m_2 r_{2\perp}^2 + m_3 r_{3\perp}^2 = (6 * 2^2 + 4 * 2^2 + 5 * 0^2)\ \mathrm{kg\ m^2} = 40\ \mathrm{kg\ m^2}$$

---

c) If the system is rotating about the y-axis with an angular acceleration of $7\ \mathrm{rad/s^2}$, calculate the torque acting on these system of particles.

Solution: $I_y = 40\ \mathrm{kg\ m^2}$; $\alpha = 7\ \mathrm{rad/s^2}$; $\tau = ?$

$$\tau = I_y\,\alpha = 40 * 7\ \mathrm{N\ m} = 280\ \mathrm{N\ m}$$

Obtaining moment of inertia of solid objects requires the use of calculus. But formulas for the moment of inertia of common shapes such as cylinder and sphere may be found in physics text books. For example the moment of inertia of a spherical object of mass M and radius R about an axis through the center of the sphere is given by $I_{sphere} = 2MR^2/5$.

8.4.2 Rotational Kinetic Energy

Let's consider an object of mass m revolving in a circular path of radius $r_\perp$ with a speed of v. This motion can be looked at as linear motion in the circular trajectory or rotational motion about the center. Its linear kinetic energy which should be equal to the rotational kinetic energy is given by $KE = mv^2/2$. But the linear speed v can be expressed in terms of the angular speed ω as $v = r_\perp \omega$; and the kinetic energy may be written as $KE = mr_\perp{}^2\omega^2/2$. The expression $mr_\perp{}^2$ is the moment of inertia, I, of the object. Therefore the rotational kinetic energy (KE_{rot}) of the object is given by the following expression.

$$KE_{rot} = I\omega^2/2$$

Example: A solid spherical object of mass *10* kg and radius *0.2* kg is rotating about an axis that passes through its center with an angular speed of *2* rad/s. Calculate its rotational kinetic energy.

Solution: $M = 10$ kg; $R = 0.2$ m; $\omega = 2$ rad/s; $KE_{rot} = ?$

$$I_{sphere} = 2MR^2/5 = 2 * 10 * 0.2^2/5 \text{ kg m}^2 = 1.6 \text{ kg m}^2$$

$$KE_{rot} = I_{sphere}\,\omega^2/2 = 1.6 * 2^2/2 \text{ J} = 3.2 \text{ J}$$

8.4.3 Kinetic Energy of a Rolling Object

A rolling object has both translational and rotational kinetic energy because it rotates as it moves. The kinetic energy of a rolling object is the sum of its translational (KE_{tra}) and rotational kinetic energy (KE_{rot}). If a rolling object is moving with a speed of v and rotating with angular speed ω, its kinetic energy (KE_{rol}) is given by

$$KE_{rol} = KE_{tra} + KE_{rot} = Mv^2/2 + I\omega^2/2$$

The translational speed, v, and rotational speed, ω, are related: $v = R\omega$. R is the radius of rotation.

Example: A sphere of mass *5* kg and radius *0.1* m is rolling in a horizontal surface with a speed of *4* m/s. Calculate its kinetic energy.

Solution: $M = 5$ kg; $R = 0.1$ m; $v = 4$ m/s; $KE_{rol} = ?$

$$I_{sphere} = 2MR^2/5 = 2 * 5 * 0.1^2 \text{ kg m}^2/5 = 0.02 \text{ kg m}^2$$

$$\omega = v/R = 4/0.1 \text{ rad/s} = 40 \text{ rad/s}$$

$$KE_{rol} = Mv^2/2 + I_{sphere}\,\omega^2/2 = 5 * 4^2/2 + 0.02 * 40^2/2 \text{ J} = 48 \text{ J}$$

Example: A spherical object of radius 0.2 m is rolling down a 10 m $30°$ inclined plane. Calculate its speed by the time it reaches the ground.

Solution: The forces acting on the object are gravity and friction. Even though friction is non-conservative, it doesn't contribute to the work done because each particle of the sphere is only instantaneously in contact with the plane. Thus the principle of conservation of mechanical energy can be applied. Let the origin of the coordinate system be fixed at the ground.

$y_f = 0$; $v_i = \omega_i = 0$; $y_i = 10 * \sin 30°$ m $= 5$ m; $R = 0.2$; $v_f = ?$

$$mv_i^2/2 + I\omega_i^2/2 + m|g|y_i = mv_f^2/2 + I\omega_f^2/2 + m|g|y_f$$

$$m|g|y_i = mv_f^2/2 + (2mR^2/5)(v_f/R)^2/2$$

$$v_f = \sqrt{(10|g|y_i/7)} = \sqrt{(10 * 9.8 * 5/7)} \text{ m/s} = 8.4 \text{ m/s}$$

8.5 Angular Momentum

Angular momentum is a physical quantity used as a measure of the rotational motion an object has. It is defined to be the product of the moment of inertia of the object and its angular speed.

$$L = I\omega$$

L is the angular momentum of an object of moment of inertia I rotating with an angular speed ω.

8.5.1 Principle of Conservation of Angular Momentum

Since $\tau_{net} = I\alpha$ and $\alpha = \Delta\omega/\Delta t$, $\tau_{net} = I\Delta\omega/\Delta t = \Delta(I\omega)/\Delta t$. Torque is equal to the rate of change of angular momentum with time.

$$\tau_{net} = \Delta L/\Delta t$$

If the net torque acting on a system is zero, it follows that $\Delta L = 0$ or $L_i = L_f$. That is, the angular momentum of the system does not change. The *Principle of conservation of angular momentum* states that if the net torque acting on a system is zero, then its angular momentum is conserved.

If

$$\tau_{net} = 0$$

Then

$$L_i = L_f$$

Orr

$$I_i\omega_i = I_f\omega_f$$

Example: A skater of moment of inertia 5 kg m^2 is revolving with an angular speed of 0.2 rad/s. She extends her hand to increase her moment of inertia to 6.2 kg m^2. Calculate her new angular speed.

Solution: Since only internal forces are involved during this change, net torque acting on the skater is zero and angular momentum is conserved.

$I_i = 5$ kg m^2; $\omega_i = 0.2$ rad/s; $I_f = 6.2$ rad/s; $\omega_f = ?$

$$I_i\,\omega_i = I_f\,\omega_f$$

$$(5\text{ kg m}^2)(0.2\text{ rad/s}) = (6.2\text{ kg m}^2)\omega_f$$

$$\omega_f = 0.16\text{ rad/s}$$

8.6 Practice Quiz 8.2

Choose the best answer. Answers can be found at the back of the book.

1. If the perpendicular distance between the axis of rotation and a particle is multiplied by a factor of *5* , then the moment of inertia of mthe particle will be multiplied by a factor of
   A. *5*
   B. *25*
   C. *0.04*
   D. *0.2*
   E. *2.236*

2. An object is said to be in translational equilibrium if
   A. it is either at rest or moving with a constant acceleration
   B. it is either at rest or moving with a constant speed
   C. it is either at rest or rotating with a constant angular acceleration.
   D. it is either at rest or moving in a straight line with a constant speed.
   E. it is either at rest or rotating with a constant angular velocity.

3. An object of mass *7.3* kg is revolving in a circular path of radius *6.4* m. Calculate its moment of inertia about an axis of rotation perpendicular to the plane of the circle passing through the center of the circle.
   A. *328.909* kg m^2
   B. *239.206* kg m^2
   C. *299.008* kg m^2
   D. *209.306* kg m^2
   E. *418.611* kg m^2

4. Three particles of masses *6.3* kg, *5.5* kg and *3.7* kg are located at the points (*9, -1*) m, (*-2, -2*) m, and (*8, 2*) m respectively. Calculate the moment of inertia of these system of particles if the axis of rotation is the x-axis.
   A. *25.86* kg m^2
   B. *43.1* kg m^2
   C. *47.41* kg m^2

D. *56.03* kg m^2
E. *34.48* kg m^2

5. Three particles of masses *5.3* kg, *10.5* kg and *1.7* kg are located at the points (*15, 0*) m, (*-5, 0*) m, and (*2, 1*) m respectively. Calculate the rotational kinetic energy of this system of particles if they are revolving around the x-axis with an angular velocity of *9* rad / s.
A. *61.965* J
B. *96.39* J
C. *68.85* J
D. *48.195* J
E. *82.62* J

6. Three particles of masses *1.3* kg, *7.5* kg and *7.7* kg are located at the points (*9, 2*) m, (*-2, -5*) m, and (*-10, 2*) m respectively. Calculate the angular momentum of this system of particles if they are revolving around the y-axis with an angular velocity of *3* rad / s.
A. *2715.9* J s
B. *3259.08* J s
C. *1901.13* J s
D. *2444.31* J s
E. *3802.26* J s

7. A spherical object of mass *4* kg and radius *0.024* m is rolling on a horizontal surface with a speed of *2.5* m / s. Calculate its kinetic energy. ($I_{spher} = 2MR^2 / 5$)
   - A. *19.25* J
   - B. *24.5* J
   - C. *21* J
   - D. *12.25* J
   - E. *17.5* J

8. A spherical object of mass *1* kg and radius *0.028* m is rolling down an inclined plane of length *5* m that makes an angle of *60* deg with the ground. Calculate its speed by the time it reaches the ground ($I_{spehre} = 2MR^2 / 5$)
   - A. *7.786* m / s
   - B. *10.122* m / s
   - C. *9.343* m / s
   - D. *10.9* m / s
   - E. *7.007* m / s

9. The angular momentum of a spherical object revolving about an axis passing through its center increased from *9* J s to *24* J s in *5* s. Calculate the torque acting on it.
   - A. *4.2* N m
   - B. *3* N m
   - C. *2.4* N m
   - D. *3.9* N m
   - E. *3.3* N m

10. Two particles of masses *10* kg and *21* kg located at $x = 1.3$ m and $x = -1.8$ m on the x-axis (connected by a rod of negligible mass) are revolving about the y-axis with an angular speed of *6.5* rad / s. If the *10* kg particle slides to the new location $x = 0.7$, Calculate the new angular speed with which the system is revolving.
    - A. *8.326* rad / s
    - B. *10.597* rad / s
    - C. *5.299* rad / s
    - D. *7.569* rad / s
    - E. *9.083* rad / s

9 Solids and Fluids

Your goals for this chapter are to learn about stress, strain, the relationship between stress and strain, fluids at rest, and fluids in motion.

9.1 Solids

Solid is a state of matter with a fixed volume and fixed shape. The effect of force on a solid is either to change motion or shape. The change of motion aspect of it has been described in previous chapters. The change of shape aspect of it will be discussed in this chapter.

9.1.1 Physics of Deformation

There are two physical quantities used to describe deformation of an object. These are stress and strain. *Stress* is a measure of a force's ability to deform an object. It is proportional to the magnitude of the force and inversely proportional to the area of the surface upon which the force is applied. It is defined to be the ratio between the force (F) and the area of the surface (A) over which the force is applied.

$$Stress = F/A$$

The unit of stress is N/m^2 which is defined to be the Pascal abbreviated as Pa. *Strain* is a physical quantity used as a measure of deformation. It is defined to be the ratio between the change and the original value. For example if the change is change in length, strain is defined to be ratio between the change in length and the original length. Since strain is ratio between the same physical quantities, it is unit-less.

9.1.2 Relationship between Stress and Strain

Stress and strain are directly proportional as stress is increased from zero to a certain value. At a certain value the proportionality between stress and strain ceases to apply. This point where the proportionality between stress and strain breaks down is called the **elastic limit**. As the stress is increased further beyond the elastic limit, at a certain value the material breaks down. This point where the material breaks down is called the breaking point of the material. The constant of proportionality between stress and strain is called modulus.

$$modulus = stress/strain$$

The unit of measurement for modulus is Pascal.

There are three kinds of stresses. These are tensile stress, shear stress and bulk stress. *Tensile stress* is a stress where the force is applied parallel to the length of the material and perpendicular to the cross-sectional area of the material. The deformation is change in length and the strain is defined to be the ratio between the change in length (ΔL) and the original length (L). The modulus associated with this kind of stress is called *Young's modulus* (Y).

$$Y = (F/A)/(\Delta L/L)$$

Example: A steel wire of length 5 m is subjected to a force of 200 N. The cross-sectional radius of the wire is 0.003 m. Young's modulus for steel is $2e10$ Pa. Calculate the change in its length.

Solution: $L = 5$ m; $F = 200$ N; $Y = 2e10$ Pa; $r = 0.003$ m; $\Delta L = ?$

$$A = \pi r^2 = 3.14 * 0.003^2 \text{ m}^2 = 0.000028 \text{ m}^2$$

$$Y = (F/A)/(\Delta L/L)$$

$$\Delta L = FL/YA = 200 * 5/(2e10 * 0.000028) \text{ m} = 0.0018 \text{ m}$$

Shear stress is a stress where the force is applied parallel to the surface. The effect of this stress is to produce deformation parallel to the surface. Its strain is defined to be the ratio between the deformation parallel to the surface (x) and the height (h) of the material in a direction perpendicular to this surface. The modulus associated with this kind of stress is called shear modulus (S).

$$S = (F/A)/(x/h)$$

Bulk stress is a stress where the stress is applied perpendicularly over the entire surface area of an object. The effect of this stress is to bring change in volume and its strain is defined to be the ratio between the change in volume (ΔV) and the original volume (V). The modulus associated with this kind of stress is called bulk modulus (B).

$$B = (-F/A)/(\Delta V/V)$$

The negative is introduced to make the bulk modulus positive, since the change in volume is negative; that is a decrease.

Example: A spherical solid aluminum ball of radius *0.002* m is subjected to a force of *2000* N applied throughout its surface area perpendicularly. The bulk modulus for aluminum is *7e10* Pa. Calculate the change in its volume.

Solution: For a sphere of radius r, the volume and surface area are given by $4\pi r^3/3$ and $4\pi r^2$ respectively.

$F = 2000$ N; $B = 7e10$ Pa; $r = 0.002$ m; $\Delta V = ?$

$$A = 4\pi r^2 = 4 * 3.14 * 0.002^2 \text{ m}^2 = 0.00005 \text{ m}^2$$

$$V = 4\pi r^3/3 = 4 * 3.14 * 0.002^3/3 \text{ m}^3 = 3.3e\text{-}8 \text{ m}^3$$

$$B = (F/A)/(\Delta V/V)$$

$$\Delta V = FV/(BA) = 2000 * 3.3e\text{-}8/(7e10 * 5e\text{-}5) \text{ m}^3 = 1.9e\text{-}11 \text{ m}^3$$

9.2 Fluids

Fluid is a state of matter with a fixed volume but not fixed shape. It takes the shape of its container. The *density* (ρ) of fluid is defined to be the ratio between its mass (m) and its volume (V).

$$\rho = m/V$$

The unit of measurement for density is kg/m^3.

9.3 Fluid Statics

Fluid statics is the study of fluids at rest. There cannot be shear stress on a fluid at rest because if there were, the molecules of the fluid would be moving. At any volume element (part) of the fluid the forces act over the entire surface area of the volume element perpendicularly. The force (F) per unit area (A) is called pressure (P).

$$P = F/A$$

Unit of measurement for pressure is Pascal. The pressure due to air molecules is called *atmospheric pressure*. The value of atmospheric pressure at sea level is equal to *1.013e5* Pa. Atmospheric pressure decreases with the increase of altitude. A unit of pressure called atm is defined to be equal to atmospheric pressure at sea level.

9.3.1 Dependence of Fluid Pressure on Depth

Let's consider a part of a fluid in rest in cylindrical shape with the base of the cylinder parallel to the surface of the fluid. Let the base area of the cylinder be A and its height h. Since the fluid is at rest, this cylinder is in equilibrium and hence the net force acting on it must be zero. The forces acting on the cylinder are its weight and the force due to pressure difference between that at the bottom surface (P) and that at the top surface (P_0). The direction of the force due to pressure difference must be upwards since the direction of weight is downwards. The force due to pressure difference is equal to $(P - P_o)A$. The weight of the cylinder is $m|g|$. Its mass is equal to the product of the density (ρ) of the fluid and its volume (V) and its volume is equal to the product of its base area (A) and its height (h). Therefore the weight of the cylinder is equal to $Ah\rho|g|$. Equating its weight to the force due to pressure difference, the following equation for the dependence of pressure on depth can be obtained.

$$P = P_o + \rho|g|h$$

P_o is pressure at the upper level and P is pressure at the lower level. h is the separation between both levels. Pressure increases with depth linearly.

Example: Calculate the pressure *5* m below the surface of an ocean. Assume the density of the ocean to be *1000* kg/m^3

Solution: The upper surface of the ocean is subjected to air molecules. Therefore the pressure at the surface of an ocean is equal to atmospheric pressure at sea level.

$$h = 5 \text{ m};\ P_o = 1.013e5 \text{ Pa};\ \rho = 1000 \text{ kg}/\text{m}^3;\ P = ?$$

$$P = P_o + \rho|g|h = (1.013e5 + 1000 * 9.8 * 5) \text{ Pa} = 150300 \text{ kg}/\text{m}^3$$

9.3.2 Measuring Pressure

Pressure is measured by a device called *manometer*. A manometer is essentially a U shaped tube filled with a fluid (most of the time mercury). From the knowledge of the pressure on one side of the U tube and the difference of the fluid levels on both sides, an unknown pressure on the other side of the tube can be calculated. There are two types of manometers. These are the closed manometer and the open manometer.

An *open manometer* is a manometer with one of the sides of the U tube open (exposed to air molecules). Thus the pressure at the surface of the fluid on this side of the tube is equal to atmospheric pressure. The gas of unknown pressure is connected to the other side of the tube and the difference between the fluid levels on both sides measured. Let atmospheric pressure be represented by P_{at} and the unknown gas pressure be denoted by P_g. If the level of the fluid on the gas side is lower than the other side, then $P_g = P$ (pressure at lower level) and $P_{at} = P_o$ (pressure at higher level). Therefore,

$$P_g = P_{at} + \rho|g|h$$

Where ρ is the density of the fluid and h is the separation between the fluid levels on both sides. Similarly if the fluid level at the air side is lower than the fluid level at the gas side, $P_g = P_o$ and $P_{at} = P$; and thus

$$P_g = P_{at} - \rho|g|h$$

Example: A gas of unknown pressure is connected to an open manometer filled with mercury. The density of mercury is $13,\ 600\ \mathrm{kg/m^3}$. Atmospheric pressure is 100 kPa.

a) If the air side fluid level is 0.02 m higher than the gas side fluid level, calculate the gas pressure.

Solution: $h = 0.02$ m; $P_{at} = 100$ kpa $= 100,\ 000$ Pa; $\rho = 13,\ 600\ \mathrm{kg/m^3}$; $p_g = ?$

$$P_g = P_{at} + \rho|g|h = (100,\ 000 + 13,\ 600 * 9.8 * 0.02)\ \mathrm{Pa} = 102665.6\ \mathrm{Pa}$$

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b) If the air side fluid level is 0.03 m below the gas side fluid level, calculate the gas pressure.

Solution: $h = 0.03$ m; $P_g = ?$

$$P_g = P_{at} - \rho|g|h = (100,\ 000 - 13,\ 600 * 9.8 * 0.03)\ \mathrm{Pa} = 96001.6\ \mathrm{Pa}$$

A closed manometer is a manometer with one side closed. The pressure on the surface of the fluid on the closed side is approximately zero because it is made to be approximately vacuum. Thus the upper level pressure is zero and the lower level pressure is the gas pressure.

$$P_g = \rho|g|h$$

Example: A gas of unknown pressure is connected to a closed manometer. The gas side fluid level is 0.08 m below the closed side fluid level. The fluid is mercury.

Solution: $h = 0.08$ m; $\rho = 13,\ 600\ \mathrm{kg/m^3}$; $p_g = ?$

$$P_g = \rho|g|h = (13,\ 600 * 9.8 * 0.08)\ \mathrm{Pa} = 10662.4\ \mathrm{Pa}$$

9.3.3 Measuring Atmospheric Pressure

Atmospheric pressure is measured by a device called *barometer*. A barometer is essentially a closed manometer. A closed tube is made to be approximately vacuum and inserted in a dish of mercury. The pressure on the closed tube is approximately zero. The mercury in the dish is exposed to atmospheric pressure. Because of the pressure difference, the mercury rises to a height h. The upper level pressure is zero and the lower level pressure is atmospheric pressure (P_{at}). Therefore,

$$P_{at} = \rho|g|h$$

Example: To what height would a mercury barometer rise at sea level?

Solution: $P_{at} = 1.013e5$ Pa; $\rho = 1.36e4$ kg / m^3; $h = ?$

$$P_{at} = \rho|g|h$$

$$h = P_{at} / (|g|\rho) = 1.013e5 / 9.8 / 1.36e4 \text{ m} = 0.76 \text{ m}$$

9.4 Practice Quiz 9.1

Choose the best answer. Answers can be found at the back of the book.

1. The ratio between the stress acting on an object and the resulting strain is called
   A. Hook's constant
   B. modulus
   C. elastic limit
   D. force constant
   E. stress constant

2. The kind of stress where the force is applied parallel to the surface is called
   A. shear stress
   B. normal stress
   C. parallel stress
   D. tensile stress
   E. bulk stress

3. An aluminum wire has a length of *8* m and a cross-sectional radius of *0.003* m. Calculate the change in its length when an object of weight *30* N hangs from the wire. (modulus = *7e10* Pa)
   A. *9.701e-5* m
   B. *7.276e-5* m
   C. *12.126e-5* m
   D. *15.764e-5* m
   E. *13.339e-5* m

4. A spherical solid aluminum ball of radius *0.05* m is compressed by a force that is applied perpendicularly throughout its entire surface area. If its volume changed by *3e-9* m^3, calculate the magnitude of the force. (Modulus = *2.5e10* Pa).
   A. *2700* N
   B. *4500* N
   C. *5400* N
   D. *5850* N
   E. *4050* N

5. A cylindrical object of height *0.095* m and radius *0.045* m is resting on the top of a table. The density of the object is *4e3* kg/m^3. Calculate the pressure exerted by the weight of the cylinder on the table.
   A. *2234.4* Pa
   B. *3724* Pa
   C. *4468.8* Pa
   D. *4096.4* Pa
   E. *2606.8* Pa

6. At what depth in an ocean would the pressure be *5* times atmospheric pressure at sea level? (Assume the density of the ocean to be *1000* kg/m^3)
   A. *41.347* m
   B. *57.886* m
   C. *28.943* m
   D. *45.482* m
   E. *33.078* m

7. In an open manometer filled with mercury (density = *13600* kg/m^3), the level of mercury column in the air side is *0.09* m higher than that in the gas side. Determine the pressure of the gas. Atmospheric pressure is *9.5e4* Pa.
   A. *14394.24* Pa
   B. *106995.2* Pa
   C. *11995.2* Pa
   D. *13194.72* Pa
   E. *83004.8* Pa

8. In an open manometer filled with mercury (density = *13600* kg/m^3), the level of mercury column in the air side is *0.03* m lower than that in the gas side. Determine the pressure of the gas. Atmospheric pressure is *9.8e4* Pa.
   A. *94001.6* Pa
   B. *3198.72* Pa
   C. *101998.4* Pa
   D. *3598.56* Pa
   E. *3998.4* Pa

9. In a closed manometer filled with mercury (density = *13600* kg/m^3), the level of mercury column in the vacuum side is *0.05* m higher than that in the gas side. Determine the pressure of the gas. Atmospheric pressure is *9.8e4* Pa.
   A. *7996.8* Pa
   B. *6664* Pa
   C. *91336* Pa
   D. *7330.4* Pa
   E. *104664* Pa

10. Determine the atmospheric pressure at a place where a mercury (density = *13600* kg/m^3) barometer rises by *0.55* m.
   A. *95295.2* Pa
   B. *73304* Pa
   C. *58643.2* Pa
   D. *43982.4* Pa
   E. *80634.4* Pa

9.4.1 Archimedes' Principle

Archimedes' principle states that an object immersed in a fluid is exerted upon by an upward buoyant force equal to the weight of the displaced fluid. The buoyant force is the force due to the difference between the upward pressure (on bottom surface) and the downward pressure (on top surface) acting on the object.

Let's consider a cylindrical object of density ρ_o height h and base area A immersed in a fluid of density ρ_f . The buoyant force (B) is equal to $(P - P_o)A$ where P is the pressure on the bottom surface and P_o is pressure at the top surface. But $P - P_o = \rho_f |g|h$. Therefore the buoyant force is given by $B = Ah\rho_f |g|h$. The product Ah is equal to the volume V of the cylinder.

$$B = V\rho_f |g|$$

Since the product $V\rho_f$ is equal to the mass of the fluid with the same volume as the object, it follows that buoyant force is equal to the weight of the displaced fluid.

An object immersed in a fluid weighs less than it does in air because of the upward buoyant force exerted by the fluid. The weight inside a fluid (W_f) is equal to the difference between the weight in air (W_a) and buoyant force.

$$W_f = W_a - B$$

Its weight in air is equal to the product of its mass (m_o) and gravitational acceleration; and its mass is equal to the product of its density and its volume.

$$W_a = \rho_o V|g|$$

Now the weight in fluid can be expressed in terms of the densities by using expressions for the weight in air and buoyant force in terms of density.

$$W_f = (\rho_o - \rho_f)V|g|$$

Example: An object of density 4000 kg/m^3 and volume of 0.002 m^3 is immersed in water (density 1000 kg/m^3).

a) Calculate the upward force exerted by the fluid on the object.

   Solution: $V = 0.002$ m^3; $\rho_f = 1000$ kg/m^3; $B = ?$

$$B = \rho_f V|g| = 1000 * 0.002 * 9.8 \text{ N} = 19.6 \text{ N}$$

---

b) Calculate its weight in air.

Solution: $\rho_o = 4000\ \mathrm{kg/m^3}$; $W_a = ?$

$$W_a = \rho_o V|g| = 4000 * 0.002 * 9.8\ \mathrm{N} = 79.4\ \mathrm{N}$$

c) Calculate its weight inside the fluid.

Solution: $W_f = ?$

$$W_f = W_a - B = (79.4 - 19.6)\ \mathrm{N} = 59.8\ \mathrm{N}$$

For a floating object, the weight of the object in air and the buoyant force exerted by the fluid must balance each other, because the object is in equilibrium. Suppose a floating object of density ρ_o and volume V floats in a fluid of density ρ_f with V_i part of its volume immersed. Its weight in air is equal to $\rho_o V|g|$ and according to Archimedes' principle (buoyant force equals weight of displaced fluid) the buoyant force is equal to $\rho_f V_i |g|$. Equating the weight in air and the buoyant force, the following equation for floating objects is obtained.

$$\rho_o / \rho_f = V_i / V$$

The following observations can be made from this equation: 1) an object whose density is smaller than the density of the fluid floats partially immersed 2) an object whose density is equal to the density of the fluid floats with all of its volume immersed. And 3) an object whose density is greater than that of the fluid sinks.

Example: An object floats in water (density $1000\ \mathrm{kg/m^3}$) with 25% of its volume immersed in the fluid. Calculate the density of the object.

Solution: $\rho_f = 1000\ \mathrm{kg/m^3}$; $V_i = 0.25V$; $\rho_o = ?$

$$\rho_o / \rho_f = V_i / V$$

$$\rho_o / (1000\ \mathrm{kg/m^3}) = 0.25V / V = 0.25$$

$$\rho_o = 250\ \mathrm{kg/m^3}$$

9.5 Fluid Dynamics

Fluid dynamics is the study of fluids in motion.

9.5.1 Continuity Equation

The *Continuity equation* is a mathematical statement of the fact that the amount of fluid that enters a tube is equal to the amount of fluid that leaves the tube in the same interval of time. Suppose fluid enters a tube of cross-sectional area A_1 with speed v_1 and leaves a tube of cross-sectional area A_2 with speed v_2. In a time interval Δt, fluid of length $v_1 \Delta t$ enters tube 1 and fluid of length $v_2 \Delta t$ leaves tube 2. In other words, in a time interval Δt, fluid of volume $A_1 v_1 \Delta t$ enters tube 1 and fluid of volume $A_2 v_2 \Delta t$ leaves tube 2. Equating these two volumes, the equation so called continuity equation is obtained.

$$A_1 v_1 = A_2 v_2$$

Example: Two tubes of cross-sectional radii of 0.02 m and 0.04 m are connected together. Water enters the first tube with a speed of 20 m / s. Calculate the speed with which it will leave the second tube.

Solution: $v_1 = 20$ m / s; $r_1 = 0.02$ m ($A_1 = \pi r_2^{\,2}$); $r_2 = 0.04$ m ($A_2 = \pi r_2^{\,2}$); $v_2 = ?$

$$A_1 v_1 = A_2 v_2$$

$$\pi r_1^{\,2} v_1 = \pi r_2^{\,2} v_2$$

$$(0.02\ \mathrm{m})^2 (20\ \mathrm{m/s}) = (0.04\ \mathrm{m})^2 v_2$$

$$v_2 = 5\ \mathrm{m/s}$$

9.5.2 Bernoulli's Equation

Let's consider two tubes, tube 1 and 2, connected together with the elevation of the first being y_1 and the elevation of the second being y_2. There are two kinds of forces acting on a fluid flowing through these tubes. These are gravity and the force due to pressure difference in the tubes. Gravity is a conservative force but the force due to pressure difference is non-conservative. The work done by a non-conservative force is equal to change in mechanical energy. If a part of the fluid of mass Δm is taken, the change in mechanical energy is $(\Delta m v_2^{\,2}/2 + \Delta m|g|y_2) - (\Delta m v_1^{\,2}/2 + \Delta m|g|y_1)$ and the work done by the force due to pressure difference is $(P_1 - P_2)\Delta V$ where ΔV is the volume of the fluid. Equating these two expressions, dividing the equation by ΔV and noting that $\Delta m/\Delta V$ is equal to the density of the fluid ρ, the following equation that is called Bernoulli's equation is obtained.

$$P_1 + \rho|g|y_1 + \rho v_1^{\,2}/2 = P_2 + \rho|g|y_2 + \rho v_2^{\,2}/2$$

Example: A tube of cross-sectional area 0.0004 m^2 is connected to a tube of cross-sectional area 0.0008 m^2. The second tube is elevated 0.01 m higher than the first tube. Water enters the first tube with a speed of 10 m / s. The pressure on the first tube is $10,\ 000$ Pa.

a) Calculate the speed of the fluid in the second tube.

   Solution: $v_1 = 10$ m / s; $A_1 = 0.0004$ m^2; $A_2 = 0.0008$ m^2; $v_2 = ?$

$$A_1 v_1 = A_2 v_2$$

$$(0.0004\ \mathrm{m}^2)(10\ \mathrm{m/s}) = (0.0008\ \mathrm{m}^2) v_2$$

$$v_2 = 5\ \mathrm{m/s}$$

---

b) Calculate the pressure on the second tube.

Solution: Let the origin of the coordinate system be fixed at the lower tube (tube 1).

$\rho = 1000\ \mathrm{kg/m^3}$; $P_1 = 10\,000\ \mathrm{Pa}$; $y_1 = 0$; $y_2 = 0.01\ \mathrm{m}$; $P_2 = ?$

$$P_1 + \rho|g|y_1 + \rho v_1^2/2 = P_2 + \rho|g|y_2 + \rho v_2^2/2$$

$(10\,000\ \mathrm{Pa}) + (1000\ \mathrm{kg/m^3})(10\ \mathrm{m/s})^2/2 = P_2 + (1000\ \mathrm{kg/m^3})(9.8\ \mathrm{m/s^2})(0.01\ \mathrm{m}) + (1000\ \mathrm{kg/m^3})(5\ \mathrm{m/s})^2/2$

$$P_2 = 47402\ \mathrm{Pa}$$

9.6 Practice Quiz 9.2

Choose the best answer. Answers can be found at the back of the book.

1. The statement 'An object immersed in a fluid is acted upon by an upward force equal to the weight of the displaced fluid.' is referred as
   A. Hook's law
   B. Charles's law
   C. Bernoulli's principle
   D. Boyle's law
   E. Archimedes' principle

2. Two tubes of different cross-sectional areas are connected together horizontally. Which of the following is a true statement about a fluid flowing through the tubes.
   A. The amount of fluid that leaves the wider tube is less than the amount of fluid that enters the narrower tube in the same interval of time.
   B. The speed of the fluid in the narrower tube is greater than the speed of the fluid in the wider tube.
   C. The pressure of the fluid in the narrower tube is greater than that in the wider tube.
   D. The speed of the fluid in the narrower tube is less than the speed of the fluid in the wider tube.
   E. The amount of fluid that leaves the wider tube is greater than the amount of fluid that enters the narrower tube in the same intervalof time.

3. An object of volume *2e-6* m^3 and density *6000* $\mathrm{kg/m}^3$ is immersed inside a fluid of density *5000* $\mathrm{kg/m}^3$. Calculate the force exerted by the fluid on the object.
   A. *0.098* N
   B. *0.137* N
   C. *0.108* N
   D. *0.127* N
   E. *0.118* N

4. An object of volume *4e-4* m^3 is immersed in a fluid. If the fluid is exerting an upward force of *0.7* N on the object, calculate the density of the fluid.
   A. *214.286* $\mathrm{kg/m}^3$
   B. *232.143* $\mathrm{kg/m}^3$
   C. *160.714* $\mathrm{kg/m}^3$
   D. *250* $\mathrm{kg/m}^3$
   E. *178.571* $\mathrm{kg/m}^3$

5. Calculate the density of an object that floats in a fluid of density *2000* kg / m ^{3} with *60* % of its volume immersed in the fluid.
   A. *1200* kg / m ^{3}
   B. *840* kg / m ^{3}
   C. *1680* kg / m ^{3}
   D. *1080* kg / m ^{3}
   E. *960* kg / m ^{3}

6. An object of volume *2e-5* m ^{3} and density *525* kg / m ^{3} is floating in a fluid of density *1250* kg m ^{3}. Calculate the volume of the object exposed above the surface of the fluid.
   A. *1.16e-5* m ^{3}
   B. *1.148e-5* m ^{3}
   C. *1.508e-5* m ^{3}
   D. *1.392e-5* m ^{3}
   E. *1.624e-5* m ^{3}

7. Tube *1* and tube *2* are connected together. A fluid flowing through these tubes has a speed of *4* m / s in tube *1* and a speed of *5* m / s in tube *2*. Calculate the ratio of the cross-sectional radius of tube *2* to the cross-sectional radius of tube *1*.
   A. *1.163*
   B. *0.716*
   C. *0.894*
   D. *0.984*
   E. *0.537*

8. A tube of cross-sectional-radius *0.05* m is connected with a tube of cross-sectional radius *0.02* m. If the speed of a fluid in the *0.05* m cross-sectional radius tube is *24* m / s, calculate the speed of the fluid in the *0.02* m cross-sectional radius tube.
   A. *135* m / s
   B. *150* m / s
   C. *90* m / s
   D. *195* m / s
   E. *180* m / s

9. Two tubes of different cross-sectional radii are connected horizontally. A fluid of density *2000* kg / m ^{3} enters the first tube with a speed of *0.25* m / s. The cross-sectional radius of the first tube is *0.05* m and that of the second tube is *0.13* m. If the pressure of the fluid in the first tube is *1e4* Pa, calculate the pressure of the fluid in the second tube.
   A. *13079.472* Pa
   B. *11067.246* Pa
   C. *12073.359* Pa
   D. *10061.132* Pa
   E. *6036.679* Pa

10. Two tubes of different cross-sectional radii are connected together with the first tube elevated *0.2* m above the second tube. The speed of a fluid of density *1750* kg/m ^{3} in the first and second tube respectively are *1.7* m / s and *0.1* m / s. If the pressure of the fluid in the first tube is *5e3* Pa, calculate the pressure of the fluid in the second tube.
   A. *14235* Pa
   B. *10950* Pa
   C. *12045* Pa
   D. *6570* Pa
   E. *13140* Pa

10 Thermal Physics

Your goals for this chapter are to learn about temperature, units of temperature, and the relationships between change of temperature and expansion of metals and gasses.

10.1 Temperature

Temperature of a substance is a measure of the average kinetic energy of the particles comprising the substance. In layman terms, it is a measure of the hotness or coldness of an object.

10.1.1 Units of Temperature

There are three units of temperature in common use. These are degree Celsius, degree Fahrenheit and degree Kelvin.

A *Degree Celsius* (°C) is defined to be $(1/100)^{th}$ of the temperature difference between the boiling temperature and the freezing temperature of water. Further, the freezing temperature of water is defined to be *0* °C. Thus, the boiling temperature of water is *100*°C. A *degree Fahrenheit* (°F) is defined to be $(1/180)^{th}$ of the difference between boiling temperature and freezing temperature of water. Further, the freezing temperature of water is defined to be *32*°F. Hence, the boiling temperature of water is *212* °F. A unit of *degree Kelvin* (°K) is defined to be equal to a unit of degree Celsius. But degree Kelvin is defined to be zero at absolute zero temperature. *Absolute zero temperature* is the lowest possible temperature where motion of the particles completely ceases. Absolute temperature occurs at about *-273* °C. Thus, degree Kelvin can be obtained by adding *273* to degree Celsius.

$$T/°\mathrm{K} = T/°\mathrm{C} + 273$$

10.1.2 Relationship between °C and °F

For a given interval of temperature, $(T - T_o)$, The ratio between a measure in °F and a measure in °C is *9/5* because *100* °C is equal to *180* °F.

$$(T - T_o)/°\mathrm{F} = (9/5)(T - T_o)/°\mathrm{C}$$

If the temperature T_o is taken to be the freezing temperature of water, then $T_o/°\mathrm{C} = 0$ °C and $T_o/°\mathrm{F} = 32$ °F.

$$T/°\mathrm{F} = (9/5)T/°\mathrm{C} + 32$$

Or

$$T/°\mathrm{C} = (5/9)(T/°\mathrm{F} - 32)$$

Example: Convert

a) 25 °C to °F

Solution: $T/°C = 25$; $T/°F = ?$

$$T/°F = (9/5)T/°C + 32 = (9/5)25 + 32 = 45$$

---

b) 50 °F to °C

Solution: $T/°F = 50$; $T/°C = ?$

$$T/°C = (5/9)(T/°F - 32) = (5/9)(50 - 32) = 10$$

---

c) 68 °F to °K

Solution: $T/°F = 68$; $T/°K = ?$

$$T/°C = (5/9)(T/°F - 32) = (5/9)(68 - 32) = 20$$

$$T/°K = T/°C + 273 = 20 + 273 = 293$$

10.1.3 Measuring Temperature

A device used to measure temperature is called thermometer. Changes in temperature are measured in terms of physical quantities whose change varies proportionally with changes of temperature such as length of metals and volume of fluids. Let the physical quantity whose change varies proportionally with temperature be denoted by L. Then the ratio between change in temperature, $(T - T_o)$, and the change in L, $(L - L_o)$, should be a constant. Let this constant be denoted by m.

$$(T - T_o)/(L - L_o) = m$$

The graph of T versus L should be a straight line with slope m. Let the T-intercept (value of T when L is zero) be denoted by b. Then the calibrating equation relating T and L should be given as follows:

$$T = mL + b$$

Calibrating a thermometer essentially means determining the values of m and b. Once the values of m and b are known the temperature can be measured by measuring L. The values of m and b can be determined from any two pairs of data relating T and L as shown in the following example.

Example: A thermometer is based on a fluid in a tube. The fluid rises by 0.03 m when the temperature is 15 °C and by 0.07 m when the temperature is 17 °C.

a) Obtain the calibrating equation for this thermometer.

Solution: $(T_1, L_1) = (15$ °C, 0.03 m); $(T_2, L_2) = (17$ °C, 0.07 m); $m = ?$; $b = ?$

$$m = (T_1 - T_1)/(L_2 - L_1) = (17 - 15)/(0.07 - 0.04)\ °\mathrm{C/m} = 50\ °\mathrm{C/m}$$

$$T_1 = mL_1 + b$$

$$b = T_1 - mL_1 = (15 - 50 * 0.03)\ °\mathrm{C} = 13.5\ °\mathrm{C}$$

Therefore the calibrating equation is

$$T = (50\ °\mathrm{C/m})L + 13.5\ °\mathrm{C}$$

---

b) Determine the reading of the thermometer, when the fluid level rises to 0.08 m.

Solution: $L = 0.08$ m; $T = ?$

$$T = (50\ °\mathrm{C/m})L + 13.5\ °\mathrm{C} = (50 * 0.08 + 13.5)\ °\mathrm{C} = 17.5\ °\mathrm{C}$$

10.2 Expansion of Metals

Substances generally expand with increase of temperature with the exception of water between 0 and 4 °C. This anomalous behavior of water is manifested by the fact that ice floats in water and is responsible for the existence of life in water.

10.2.1 Linear Expansion of metals

The expansion of a metal with increase of temperature is directly proportional to the change in temperature as well as to the original length of the metal.

$$\Delta L = \alpha L_o\, \Delta T$$

$\Delta L = L - L_o$ is change in length corresponding to a change in temperature $\Delta T = T - T_o$. L_o is the original length. α is a material constant called temperature coefficient of linear expansion. Its unit is $1/°\mathrm{C}$. An expression for the length after expansion can be obtained by separating L from the equation for ΔL.

$$L = L_o\, \{1 + \alpha(T - T_o)\}$$

Example: A steel rod has a length of *2* m. The temperature coefficient of linear expansion for steel is *11e-6/*°C.

a) By how much would its length change when its temperature changes by *50* °C

Solution: $\Delta T = 50$ °C; $L_o = 2$ m; $\alpha = 11e\text{-}6/$°C; $\Delta L = ?$

$$\Delta L = \alpha L_o \Delta T = 11e\text{-}6 * 2 * 50 \text{ m} = 0.0011 \text{ m}$$

---

b) Calculate its length if its temperature is increased from *20* °C to *80* °C.

Solution: $T_o = 20$ °C; $T = 80$ °C; $L = ?$

$$L = L_o \{ 1 + \alpha(T - T_o) \} = 2\{ 1 + 11e\text{-}6\ (80 - 20)\} \text{ m} = 2.0013 \text{ m}$$

10.2.2 Areal Expansion of Metals

Areal expansion is two dimensional expansion. Without loss of generality, let's consider the expansion of a square metal plate of length L_o. When the plate is heated, both dimensions of the plate will expand linearly. It will expand into a square plate of length $L_o + \Delta L = L_o + \alpha L_o \Delta T$. The new area is $A = (L_o + \alpha L_o \Delta T)^2 = L_o^2 + 2L_o^2 \alpha\Delta T + (L_o \alpha\Delta T)^2$. The last term is too small to be considered because α^2 is much smaller than one. Noting that the original area $A_o = L_o^2$, the following expression for the area after expansion is obtained.

$$A = A_o\{ 1 + 2\alpha(T - T_o)\}$$

An equation relating the change in area $\Delta A = A - A_o$ and the change in temperature $\Delta T = T - T_o$ can be obtained by rearranging this equation.

$$\Delta A = 2\alpha A_o \Delta T$$

The expression 2α is called the temperature coefficient of areal expansion and denoted by γ.

Example: A circular steel plate has a radius of 0.1 m. Its temperature coefficient of linear expansion is $11e\text{-}6/$°C. Its temperature is increased from 20 °C to 100 °C. Calculate its new area.

Solution: $r_o = 0.1$ ($A_o = \pi r_o^{\,2}$); $T_o = 20$ °C; $T = 100$ °C; $\alpha = 11e\text{-}6/$°C; $A = ?$

$$A_o = \pi r_o^{\,2} = 3.14 * (0.1)^2 = 0.0314 \text{ m}^2$$

$$A = A_o\{ 1 + 2\alpha(T - T_o)\} = 0.0314\{ 1 + 2 * 11e\text{-}6(100 - 20)\} \text{ m}^2 = 0.0315 \text{ m}^2$$

10.2.3 Volume Expansion of Metals

Volume expansion is a three dimensional expansion. Each dimension will expand linearly. Without loss of generality, let's consider the expansion of a cube of length L_o. After being heated the cube will expand into a cube of length $L_o + \alpha L_o \Delta T$ and the new volume would be $V = L^3 = (L_o + \alpha L_o \Delta T)^3 = L_o^{\,3} + 3L_o^{\,3} \alpha\Delta T + 3L_o^{\,3} \alpha^2\Delta T^2 + (L_o \alpha\Delta T)^3$. The last two terms are too small to be considered because α^2 and α^3 are much smaller than one. Noting that the original volume is $V_o = L_o^{\,3}$, the following expression for the volume after expansion is obtained.

$$V = V_o\{ 1 + 3\alpha(T - T_o)\}$$

An equation between change in the volume $\Delta V = V - V_o$ and change in temperature $\Delta T = T - T_o$ can be obtained by rearranging this equation.

$$\Delta V = 3\alpha V_o \, \Delta T$$

The expression 3α is called the temperature coefficient of volume expansion of the material and denoted by β.

Example: A spherical metal ball of radius *0.02* m is made up of steel. The temperature coefficient of temperature expansion for steel is *11e-6*. By how much would its volume change, when its temperature changes by *100* °C?

Solution: $r_o = 0.02$ m ($V_o = 4\pi r_o^{\,3}/3$); $\Delta T = 100$ °C; $\alpha = 11e\text{-}6/$°C; $\Delta V = ?$

$$V_o = 4\pi r_o^{\,3}/3 = 4 * 3.14 * (0.02)^3/3 \text{ m}^3 = 0.000033 \text{ m}^3$$

$$\Delta V = 3\alpha V_o \, \Delta T = 3 * 11e\text{-}6 * 0.000033 * 100 \text{ m}^3 = 1.1e\text{-}7 \text{ m}^3$$

10.3 Practice Quiz 10.1

Choose the best answer. Answers can be found at the back of the book.

1. Which of the following is a correct statement?
   A. One unit of degree Fahrenheit is equal to (*1/100*)th of the temperature difference between boilingand freezing point of water.
   B. One unit of degree Celsius is equal to one unit of degree Kelvin.
   C. One unit of degree Celsius is equal to (*1/180*)th of the temperature difference between boilingand freezing point of water.
   D. One unit of degree Fahrenheit is equal to one unit of degree Kelvin.
   E. The SI unit of temperature is the degree Celsius.

2. On what range of temperature does water contract with increase of temperature instead of expanding?
   A. between *-4* and *4* degree Celsius
   B. between *0* and *-4* degree Fahrenheit
   C. between *0* and *4* degree Celsius
   D. between *0* and *-4* degree Celsius
   E. between *0* and *4* degree Fahrenheit

3. *10* °F is equal to
   - A. *-12.222* °C
   - B. *-17.111* °C
   - C. *-9.778* °C
   - D. *-8.556* °C
   - E. *-11* °C

4. *104* °F is equal to
   - A. *406.9* °K
   - B. *187.8* °K
   - C. *313* °K
   - D. *375.6* °K
   - E. *344.3* °K

5. A thermometer is based on the change of volume with change in temperature of a fluid in a tube. The fluid rises to a level of *0.05* m when the temperature is *10* ° C and to a level of *0.1* m when the temperature is *17.5* ° C. Obtain the calibrating equation; that is a formula for the temperature (*T*) in terms of the fluid level (*L*).
   - A. *T = 150 * L + 2.75*
   - B. *T = 120 * L + 1.5*
   - C. *T = 150 * L + 2.5*
   - D. *T = 120 * L + 2.5*
   - E. *T = 195 * L + 2.75*

6. A thermometer is based on the change of volume with change in temperature of a fluid in a tube. The fluid rises to a level of *0.035* m when the temperature is *10* ° C and to a level of *0.13* m when the temperature is *18.5* ° C. Determine the reading of the thermometer when the fluid rises to a level of *0.22* m.
   - A. *26.552* °C
   - B. *18.587* °C
   - C. *37.173* °C
   - D. *31.863* °C
   - E. *23.897* °C

7. Calculate the change in length of an aluminum rod of length *5* m when its temperature increases by *110* °C. (Coefficient of linear expansion of aluminum is *2.4e-5* / °C)
   - A. *0.792e-2* m
   - B. *1.584e-2* m
   - C. *1.188e-2* m
   - D. *0.924e-2* m
   - E. *1.32e-2* m

8. An aluminum rod has a length of *5* m at a temperature of *20* °C. What should its temperature be if its length is to increase by *0.004* m. (Coefficient of linear expansion for aluminum is *2.4e-5* / °C).
   - A. *48* °C
   - B. *42.667* °C
   - C. *32* °C
   - D. *37.333* °C
   - E. *53.333* °C

9. A circular plate made of steel has a radius of *0.1* m at a temperature of *20* °C. Calculate its radius at a temperature of *190* °C. (Coefficient of linear expansion of steel is *1.1e-5* ⁄ °C)
   A. *0.1004* m
   B. *0.1006* m
   C. *0.1002* m
   D. *0.1* m
   E. *0.1007* m

10. By how much should the temperature of an aluminum sample change if its volume is to change by *3.25* % of its volume? (Coefficient of linear expansion of aluminum is *2.4e-5* ⁄ °C)
    A. *496.528* °C
    B. *451.389* °C
    C. *361.111* °C
    D. *270.833* °C
    E. *315.972* °C

10.4 Expansion of Gasses

Gas is a state of matter with no fixed shape and no fixed volume. It takes the volume of its container. There are two ways by which the dynamics of gasses can be described. These are macroscopic and microscopic description. *Macroscopic description* is description based on the average properties of the particles comprising the gas. *Microscopic description* is a statistical description of the properties of the particles comprising the gas.

10.4.1 Macroscopic Description

The state variables used for the macroscopic description of gasses are temperature (T), pressure (P) and volume (V). Volume is the volume of the container of the gas. Pressure is the average force per unit area exerted on the container by the particles of the gas. Temperature is the average of the kinetic energies of the particles. These three variables are related by a law called the combined gas law.

10.4.2 The Combined Gas Law

The *combined gas law* states that the volume of a gas is directly proportional to its temperature (in °K) and inversely proportional to its pressure. Mathematically, this means the state of the gas might change but the ratio between the product of volume and pressure, and temperature is a constant. If the state of a gas changes from the state (V_1, T_1, P_1) to the state (V_2,T_2,P_2), then the following equation holds.

$$V_1 P_1 / T_1 = V_2 P_2 / T_2$$

The temperature must be in °K when using this equation. °C and °F can't be used. There are two special cases of this law called Boyle's law and Charles Law.

Boyle's law states that if temperature is kept constant, then the volume of a gas is inversely proportional to its pressure. That is, the product of volume and pressure of a gas remains constant.

$$P_1 V_1 = P_2 V_2$$

Charles' law states that if pressure of a gas is kept constant, then the volume and the temperature of the gas are directly proportional. That is, the ratio between volume and temperature is a constant.

$$V_1/T_1 = V_2/T_2$$

The freezing temperature of water (*0* °C) and the atmospheric pressure at sea level (*101300* Pa) are commonly referred as standard temperature and standard pressure respectively. The abbreviation STP stands for standard temperature and pressure.

Example: A gas in a balloon has a volume of $20\ \text{cm}^3$ when the temperature is 25 °C and the pressure is 50 kPa. Calculate its volume at STP.

Solution: $V_1 = 20\ \text{cm}^3$; $T_1 = 25$ °C ($T/°\text{K} = T/°\text{C} + 273$); $P_1 = 50,000$ Pa; $T_2 = 273$ °K; $P_2 = 101,300$ Pa; $V_2 = ?$

$$T_1/°\text{K} = T_1/°\text{C} + 273 = 25 + 273 = 298$$

$$T_1 = 298\ °\text{K}$$

$$V_1 P_1/T_1 = V_2 P_2/T_2$$

$$V_2 = V_1 P_1 T_2/(T_1 P_2) = 20 * 50,000 * 273/(298 * 101,300)\ \text{cm}^3 = 9\ \text{cm}^3$$

10.4.3 Brief Review of Chemistry

The mass of atoms is measured by a unit of mass called atomic mass unit, abbreviated as u. *Atomic mass unit* is defined to be $(1/12)^{\text{th}}$ of the mass of a carbon atom. The mass of the atom of an element measured in atomic mass unit is called *atomic mass* (M) of the element. For example the mass of a carbon atom (M_C) is 12 u and the mass of an oxygen atom (M_O) is 16 u. *Molecular mass* of a compound is defined to be the sum of the atomic masses of the atoms in the chemical formula of the compound. For example the molecular mass of carbon di-oxide whose chemical formula is CO_2 is the sum of the atomic mass of a carbon atom and twice the atomic mass of oxygen: $M_{CO2} = M_C + 2M_O = (12 + 2 * 16)$ u $= 44$ u. The *gram molecular weight* (M_g) of a compound is defined to be its molecular mass expressed in grams (That is $M_g = (M/\text{u})$ g). For example the gram molecular weight of carbon di-oxide is 44 g because its molecular mass is 44 u. One gram molecular weight of any substance contains $6.02e23$ molecules. The number $6.02e23$ is called Avogadro's number and denoted as N_A.

$$N_A = 6.02e23$$

One gram molecular weight of a substance is also called one *mole* of the substance. Thus the mass of one mole of a substance is equal to gram molecular weight of the substance and there are Avogadro number of molecules in one mole of a substance. The number of moles, n, in a sample of mass m (in grams) may be obtained by dividing the mass of the sample in grams by the molecular weight of the sample, M_g.

$$n = m/M_g$$

Also, the number of moles in a substance can be obtained as a ratio between the number of molecules in the sample, N, and Avogadro number.

$$n = N/N_A$$

Example: A sample of water has a mass of *36* g. The chemical formula of water is H_2O. Atomic masses of hydrogen and oxygen are *1* u and *16* u respectively.

a) How many moles are there in the sample?

Solution: $m = 36$ g; $M_g = (2*1 + 16)$ g $= 18$ g; $n = ?$

$$n = m/M_g = 36/18 = 2$$

---

b) How many water molecules are there in the sample?

Solution: $N = ?$

$$n = N/N_A$$

$$N = nN_A = 2 * 6.02e23 = 12.04e23$$

10.4.4 The Ideal Gas Equation

The combined gas law was stated for a fixed amount of gas. For a situation where the amount of gas also might change, the proportionality between volume and number of moles should also be included. The combined gas law should be restated to say the volume of a gas is directly proportional to temperature and number of moles, and inversely proportional to pressure. Mathematically, this means the ratio between the product of volume and pressure to the product of number of moles and temperature is a constant (PV/nT = constant). This constant is a universal constant called universal gas constant, denoted by R.

$$PV = nRT$$

This is the equation known as the ideal gas equation. The value of R is *8.3* J/°K/mole.

$$R = 8.3 \text{ J/°K/mole}$$

The ideal gas equation is applicable to real gasses only approximately. It applies exactly only for the ideal case where the interaction energy between the molecules can be neglected completely and the energy of the molecules can be assumed to be purely kinetic energy. In using this equation, SI units (That is m^3 for volume, Pa for pressure and °k for temperature) should be used. But Liter (Abbreviated as L) is a very common unit of volume. If Liter is used for volume, then kilo Pascal (kPa) should be used for pressure. The unit for temperature must be in °K. °C and °F cannot be used.

Example: Calculate the volume of one mole of any gas at STP in liters.

Solution: $n = 1$; $T = 273$ °K; $P = 101$ Pa; $V = ?$

$$PV = nRT$$

$$V = nRT/P = 1 * 8.3 * 273/101 \text{ L} = 22.4 \text{ L}$$

Example: *40* L of carbon di-oxide gas is at STP.

a) Calculate the number of moles in this sample.

Solution: $V = 40$ L; $T = 273$ °K; $P = 101$ kPa; $n = ?$

$$PV = nRT$$

$$n = PV/(RT) = 101 * 40/(8.3 * 273) = 1.8$$

---

b) How many grams are there in this sample

Solution: $M_g = (12 + 2*16)$ g $= 44$ g; $m = ?$

$$n = m/M_g$$

$$m = nM = 1.8 * 44 \, g = 79.2 \, g$$

The ideal gas equation can also be expressed in terms of the number of molecules instead of the number of moles. Replacing n in the ideal equation by N/N_A, The equation $PV = NRT/N_A$. The ratio R/N_A is a constant called Boltzmann's constant and is denoted by κ_β.

$$PV = N\kappa_\beta \, T$$

The value of Boltzmann's constant is *1.38e-23* J/°K/mole.

$$\kappa_\beta = 1.38e\text{-}23 \text{ J/°K/mole}$$

Example: A gas has a volume of $0.0002\ \text{m}^3$ at a temperature of $30\ °\text{C}$ and a pressure of 20 kPa. How many molecules are there in the sample?

Solution: $V = 0.0002\ \text{m}^3$; $T = (30 + 273)\ °\text{K} = 303\ °\text{K}$; $P = 20\ \text{kPa} = 20,000\ \text{Pa}$; $N = ?$

$$PV = N\kappa_\beta T$$

$$N = PV/\kappa_\beta T = 20,000 * 0.0002/(1.38e\text{-}23 * 303) = 1e{+}21$$

10.4.5 Energy of Ideal Gas Molecules

An ideal gas is a gas where the interaction energy between the molecules is neglected and the energy is purely kinetic energy. This energy is directly proportional to temperature (in °K) and depends on the number of degrees of freedom of the molecules comprising the gas. Number of degrees of freedom is equal to the number of different ways by which the particles can acquire energy. The random motion of ideal gas molecules can be decomposed into motion along the x-axis, y-axis and z-axis. That is the number of degrees of freedom of the particles is three. The energy per particle per degree of freedom is equal to $\kappa_\beta T/2$. Therefore, since a particle of an ideal gas has three degrees of freedom, the energy of one particle of an ideal gas is three times the energy per degree of freedom.

$$E = 3\kappa_\beta T/2$$

E is the energy of one molecule of an ideal gas at a temperature T (in °K). The total energy (E_T) of the sample is obtained by multiplying the energy of one particle by the number of molecules (N) in the sample.

$$E_T = 3N\kappa_\beta\, T/2$$

The total energy also can be expressed in terms of the number of moles in the sample. Replacing κ_β by R/N_A and noting that the ratio N/N_A is equal to the number of moles n, the following alternative expression for the total energy can be obtained.

$$E_T = 3nRT/2$$

Example: *10* g of hydrogen gas sample (H_2) is at a temperature of *20* °C. Atomic mass of hydrogen atom is *1* u

a) Calculate the energy of one molecule.

Solution: $T = (20 + 273)$ °K $= 293$ °K; $E = ?$

$$E = 3\kappa_\beta\, T/2 = 3 * 1.38e\text{-}23 * 293/2 = 6.1e\text{-}21 \text{ J}$$

---

b. Calculate the total energy of the sample.

Solution: $m = 10$ g; $M_g = 2 * 1$ g $= 2$ g; $E_T = ?$

$$n = m/M_g = 10/2 = 5$$

$$E_T = 3nRT/2 = 3 * 5 * 8.3 * 293/2 = 6.1e3 \text{ J}$$

10.4.6 RMS speed of Ideal gas molecules

The molecules of a gas are moving randomly with all kinds of speeds. The root mean square speed abbreviated as RMS speed is defined to be the square root of the average of the squares of the speeds of all the particles. The energy of one particle of an ideal gas may also be obtained as the kinetic energy of a particle whose speed is the RMS speed. Thus, if the mass of one molecule is denoted as m_o, the equation $m_o\, v_{RMS}{}^2/2 = 3\kappa_\beta\, T/2$ holds; and it follows that

$$v_{RMS} = \sqrt{(3\kappa_\beta\, T/m_o)}$$

m_0 (in grams) can be obtained by dividing the gram molecular weight by Avogadro number, since there are Avogadro number of particles in one gram molecular weight. It needs to be in kg though. m_o in kg may be given as follows.

$$m_o = M_{kg} / (1000 N_A)$$

Where M_{kg} is molecular mass expressed in kg (That is $M_{kg} = (M / \text{u})$ kg). Replacing κ_β by R / N_A, the following alternative expression for the RMS speed can be obtained.

$$v_{RMS} = \sqrt{(3000RT / M_{kg})}$$

Example: Calculate the RMS speed of a sample of hydrogen gas (H_2) at a temperature of *100* °K.

Solution: $T = 100$ °K; $M_{kg} = 2$ kg; $v_{RMS} = ?$

$$v_{RMS} = \sqrt{(3000RT / M_{kg})} = \sqrt{(3000 * 8.3 * 100 / 2)}\ \text{m/s} = 1115.8\ \text{m/s}$$

10.5 Practice Quiz 10.2

Choose the best answer. Answers can be found at the back of the book.

1. The combined gas law states that the volume of an ideal gas is
   A. directly proportional to its temperature and inversely proportional to its pressure.
   B. directly proportional to its temperature and pressure
   C. directly proportional to its pressure and inversely proportional to its temperature
   D. inversely proportional to its pressure and temperature
   E. is directly proportional to its temperature if its pressure is kept constant.

2. Atomic mass unit is defined to be equal to
   A. the mass of a neutron.
   B. $(1/16)^{th}$ of the mass of an oxygen atom.
   C. the mass of a hydrogen atom
   D. $(1/12)^{th}$ of the mass of a carbon atom.
   E. the mass of a proton.

3. A gas has a volume of *70* L at a pressure of *100* kPa and a temperature of *50* °C. Calculate its volume at a pressure of *15* kPa and a temperature of *15* °C.
   A. *416.099* L
   B. *374.489* L
   C. *291.269* L
   D. *499.319* L
   E. *582.539* L

4. A gas in a balloon has a volume of *100* L at a temperature of *100* °C. Calculate its temperature when its volume is changed to *45* L at constant pressure.
   A. *-63.09* °C
   B. *-115.665* °C
   C. *-73.605* °C
   D. *-105.15* °C
   E. *-126.18* °C

5. Calculate the temperature of *6* moles of a gas that occupies a volume of *70* L at a pressure of *120* kPa.
   A. *-135.623* °C
   B. *-104.325* °C
   C. *-93.893* °C
   D. *-62.595* °C
   E. *-125.19* °C

6. Calculate the volume of *90* g of di-atomic oxygen (O_2) gas at STP (atomic mass of oxygen is *16* u).
   A. *37.746* L
   B. *81.784* L
   C. *62.911* L
   D. *75.493* L
   E. *69.202* L

7. Calculate the volume of a sample of a gas that contains *6e23* molecules at a pressure of *80* kPa and a temperature *100* °C.
   A. *30.884* L
   B. *42.466* L
   C. *38.606* L
   D. *50.187* L
   E. *54.048* L

8. A *10* g sample of di-atomic hydrogen (H_2) gas is at a temperature of *40* °C. Calculate the average energy (translational kinetic energy) of one hydrogen molecule of the sample.
   A. *9.071e-21* J
   B. *6.479e-21* J
   C. *5.831e-21* J
   D. *4.535e-21* J
   E. *5.183e-21* J

9. A *40* g sample of di-atomic oxygen (O_2) gas is at a temperature of *100* °C. Calculate the total energy (translational kinetic energy) of the sample. (Atomic mass of oxygen is *16* u.)
   A. *4643.85* J
   B. *6385.294* J
   C. *5804.813* J
   D. *8126.738* J
   E. *4063.369* J

10. An *80* g sample of di-atomic hydrogen (H_2) gas is at a temperature of *30* °C. Calculate the RMS (root mean square) speed of the molecules of the sample. (Atomic mass of hydrogen is *1* u.)
   A. *2524.93* m / s
   B. *1942.254* m / s
   C. *1165.352* m / s
   D. *1748.028* m / s
   E. *1553.803* m / s

11 Heat

Your goals for this chapter are to learn about, heat, relationship between heat and change of temperature, phase changes, and ways of transfer of heat.

Heat is energy in transit from an object at higher temperature to an object at lower temperature. The SI unit of measurement for heat is the Joule. Another common unit of measurement of heat is the Calorie, abbreviated as Cal. A Calorie is defined to be equal to the amount of heat energy required to raise the temperature of one gram of water by one °C. It is equivalent to *4.18* J.

11.1 Heat and Change of Temperature

For a given substance, amount of heat supplied is directly proportional to the change of its temperature. For a given change in temperature, amount of heat supplied is directly proportional to mass.

$$Q = mc\Delta T$$

Q stands for the amount of heat supplied (lost). c is a material constant called specific heat capacity of the substance. *Specific heat capacity* of a substance is defined to be the amount of heat energy required to raise the temperature of one kilo gram of the substance by one °C. The unit of measurement for specific heat capacity is J/kg/°C. For example, *4180* J of energy is required to raise the temperature of one kilo gram of water by one °C. Therefore specific heat capacity of water is *4180* J/kg/°C.

Example: Calculate the amount of heat required to increase the temperature of *4* kg of water from *23* °C to *30* °C.

Solution: $m = 4$ kg; $T_i = 23$ °C; $T_f = 30$ °C; $c = 4180$ J/kg/°C; $Q = ?$

$$Q = mc(T_f - T_i) = 4 * 4180 * (30 - 23) \text{ J} = 117040 \text{ J}$$

11.2 Mixtures

When a hot substance is mixed with a cold substance, heat will flow from the hot substance to the cold substance until both of them have the same temperature. This temperature is called equilibrium temperature. From the principle of conservation of energy, the amount of heat lost (Q_h) by the hot object is equal to the amount of heat gained (Q_c) by the cold object.

$$Q_c = -Q_h$$

The negative is required because loss is taken to be negative and gain is taken to be positive. Suppose a hot substance of mass m_h, specific heat capacity c_h and temperature T_h is mixed with a colder substance of mass m_c, specific heat capacity c_c and temperature T_c and the final equilibrium temperature is found to be T_f. Then the heat lost by the hot substance is $Q_h = m_h\, c_h\, (T_f - T_h)$ and the heat gained by the colder substance is $Q_c = m_c\, c_c\, (T_f - T_c)$. Applying the principle of conservation of energy, the following equation for mixtures is obtained.

$$m_c\, c_c\, (T_f - T_c) = -m_h\, c_h\, (T_f - T_h)$$

Example: *0.05* kg of aluminum metal at a temperature of *100* °C is mixed with *0.15* kg of water at a temperature of *22* °C. Calculate the equilibrium temperature of the mixture. Specific heat capacity for aluminum is *900* J/kg/°C

Solution: $m_h = 0.05$ kg; $c_h = 900$ J/kg/°C; $T_h = 100$ °C; $m_c = 0.15$ kg; $c_c = 4180$ J/kg/°C; $T_c = 22$ °C; $T_f = ?$

$$m_c\, c_c\, (T_f - T_c) = -m_h\, c_h\, (T_f - T_h)$$

$$m_c\, c_c\, T_f - m_c\, c_c\, T_c = -m_h\, c_h\, T_f + m_h\, c_h\, T_h$$

$$(m_c\, c_c + m_h\, c_h)T_f = m_c\, c_c\, T_c + m_h\, c_h\, T_h$$

$$T_f = (m_h\, c_c\, T_c + m_h\, c_h\, T_h)/(m_c\, c_c + m_h\, c_h) = (0.15 * 4180 * 22 + 0.05 * 900 * 100)/(0.15 * 4180 + 0.05 * 900)\ °C = 27.2\ °C$$

11.3 Heat and Phase Changes

There are three phases of matter. These are solid, liquid and gas phase. Solid phase is a phase with a fixed shape and a fixed volume. Liquid phase is a phase with a fixed volume but not fixed shape. Gas phase is a phase with no fixed volume and no fixed shape. One phase can be changed to another by the supply or loss of heat. During a phase change, the amount of heat supplied is used to break the bonds between the molecules and not to increase the kinetic energy of the molecules. Thus, temperature remains constant during a phase change. For example, during the melting process, the temperature remains constant at the melting temperature until the substance melts completely.

11.3.1 Latent Heat of Fusion

Amount of heat required to melt (freeze) a substance at the melting (freezing) temperature is directly proportional to the mass of the substance. The amount of heat required to melt (freeze) one kilogram of a substance at the melting (freezing) temperature is called the *Latent heat of fusion* of the substance. The unit of measurement for latent heat of fusion is J / kg. For example, *334000* J of energy is required to melt (freeze) ice (water). Thus Latent heat of fusion for water is *334000* J / kg.

$$H_f = mL_f$$

H_f is the amount of heat required to melt a substance of mass m at the melting temperature. L_f is the latent heat of fusion of the substance.

Example: Calculate the amount of heat required to melt *0.2* kg of ice.

Solution: $m = 0.2$ kg; $L_f = 334000$ J / kg; $H_f = ?$

$$H_f = mL_f = 0.2 * 334000 \text{ J} = 66800 \text{ J}$$

11.3.2 Latent Heat of Vaporization

The amount of heat required to vaporize (condense) a substance at the boiling temperature is proportional to the mass of the substance. The amount of heat required to vaporize (condense) one kilogram of a substance at the boiling temperature is called the *latent heat of vaporization* of the substance. The unit of measurement for the latent heat of vaporization is J / kg. For example *2.26e6* J of energy is required to vaporize (condense) water (steam). Therefore Latent heat of vaporization of water is *2.26e6* J / kg.

$$H_v = mL_v$$

H_v is the amount of heat required to melt a substance of mass m at the boiling temperature. L_v is the latent of heat of vaporization of the substance.

Example: Calculate the amount of heat required to vaporize *0.03* kg of water.

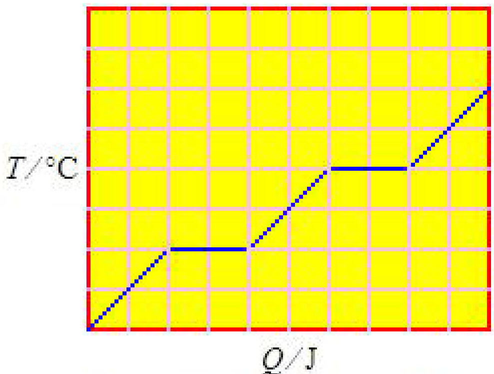
Solution: $m = 0.03$ kg; $L_v = 2.26e6$ J / kg; $H_v = ?$

$$H_v = mL_v = 0.03 * 2.26e6 \text{ J} = 67800 \text{ J}$$

11.3.3 The Graph of Temperature versus Heat

As a solid is supplied with heat, its temperature will increase linearly with the heat supplied until the melting temperature is reached. Once the melting temperature is reached, the heat energy is used to break the bonds among the solid molecules to form liquid and not to increase the kinetic energy of the molecules. Thus the temperature remains constant during the melting process. Once the solid has completely melted, once again the heat energy is used to increase the kinetic energy of the liquid molecules and the temperature of the liquid increases linearly with the amount of heat supplied until the boiling temperature is reached. Once the boiling temperature is reached, the heat energy is used to break the bonds among the liquid molecules to form gas and not to increase the kinetic energy of the molecules. As a result, the temperature remains constant at the boiling temperature during the boiling process. After all the liquid has vaporized, the heat energy is used to increase the kinetic energy of the gas molecules and the temperature increases linearly with the amount of heat supplied.

The following is a general representation for the graph of temperature versus heat for an arbitrary substance. The temperature at the lower horizontal line of the graph is the melting temperature (T_m) of the substance. The temperature at the upper horizontal line is the boiling temperature (T_b) of the substance. The phase at the lower slanted line is pure solid. At this interval the amount of heat and change in temperature are related by $Q = mc_s\ \Delta T$ where c_s is the specific heat capacity of the solid phase. At the lower horizontal line the solid is melting and the phase is a mixture of solid and liquid. On this interval the amount of heat and the change in temperature are related by $Q = mL_f$. At the middle slanted line the phase is pure liquid. The amount of heat and the change in temperature are related by $Q = mc_l\ \Delta T$ where c_l is the specific heat capacity of the liquid phase. At the upper horizontal line the liquid is vaporizing and the phase is a mixture of liquid and gas phases. At this interval, the amount of heat and the change in temperature are related by $Q = mL_v$. At the upper slanted line, the phase pure gas. At this stage the amount of heat and change in temperature are related by $Q = mc_g\ \Delta T$ where c_g is the specific heat of the gas phase.



The graph of temperature versus heat

Figure 11.1

To calculate the amount of heat for a change of temperature encompassing different stages, the amount of heat for each stage should be calculated first and then added. For example, to calculate the amount of heat required to convert ice at *-10* °C to water at *80* °C, first the amount of heat needed to convert ice at *-10* °C to ice at *0* °C, the amount of heat needed to melt the ice completely and the amount of heat needed to convert water at *0* °C to *80* °C should be calculated separately and then added.

Example: Specific heat capacities of water, ice and steam are *4180* J / kg / °C, *2090* J / kg / °C, *2010* J / kg / °C respectively. Latent heats of fusion and vaporization for water are *334000* J / kg and *2.26e6* J / kg respectively.

a) Calculate the amount of heat required to convert *0.02* ice at *-20* °C to water at *0* °C.

Solution: The amount of heat needed to convert ice at *-20* °C to ice at *0* °C and the amount of heat needed to melt ice should be obtained and added.

$m = 0.02$ kg; $c_s = 4180$ J / kg / °C; $c_l = 2090$ J / kg / °C; $c_g = 2010$ J / kg / °C; $T_m = 0$ °C; $T_i = -20$°C; T_f; $L_f = 334000$ J / kg; $Q = ?$

$$Q = mc_s\,(T_m - T_i) + mL_f + mc_l\,(T_f - T_m) = \{0.02 * 2090 * (0 - (-20)) + 0.02 * 334000 + 0.02 * 4180 * (80 - 0)\}\ \text{J} = 4848\ \text{J}$$

---

b) Calculate the amount of heat required to convert *0.004* kg of water at *20* °C to steam at *105* °C.

Solution: The amount of heat needed to convert water at *20* °C to water at water at *100* °C, amount of heat needed to vaporize it completely and amount of heat needed to change the temperature of the steam from *100*°C to *105* °C should be calculated separately and then added.

$m = 0.004$ kg; $c_g = 2010$ J / kg / °C; $L_v = 2.26e6$ J / kg; $T_b = 100$ °C; $T_i = 20$ °C; $T_f = 105$ °C; $Q = ?$

$$Q = mc_l\,(T_b - T_i) + mL_v + mc_g\,(T_5 - T_b) = \{0.004 * 4180 * (100 - 20) + 0.004 * 2.26e6 + 0.004 * 2010 * (105 - 100)\}\ \text{J} = 10418\ \text{J}$$

11.4 Practice Quiz 11.1

Choose the best answer. Answers can be found at the back of the book.

1. Heat is
   A. a measure of the average kinetic energy of the particles of a system.
   B. the total potential energy of the particles of a system.
   C. the total kinetic energy of the particles of a system.
   D. energy in transit from high temperature system to a low temperature system.
   E. the total energy of the particles of a system.

2. What is the SI unit of measurement for the specific heat capacity of a substance?
   A. (degree Celsius) / kilogram
   B. (degree Celsius) / Joule
   C. Joule / kilogram / (degree Celsius)
   D. Joule / (degree Celsius)
   E. Joule / kilogram

3. When a sample of lead (specific heat *130* J/kg/°C) is supplied with a *14000* J of heat energy, its temperature increased from *14* °C to *92* °C. Calculate the mass of the sample.
   A. *1.105* kg
   B. *1.933* kg
   C. *1.519* kg
   D. *0.966* kg
   E. *1.381* kg

4. A bullet of mass *0.06* kg was fired into an aluminum metal (specific heat capacity *900* J/kg/°C) of mass *2.5* kg with a speed of *850* m/s. If half of the kinetic energy of the bullet is transferred to the aluminum metal as heat energy, by how much would the temperature of the aluminum metal increase?
   A. *5.78* °C
   B. *6.262* °C
   C. *4.817* °C
   D. *4.335* °C
   E. *2.89* °C

5. A *0.75* kg of aluminum metal (specific heat capacity *900* J / kg / °C) at a temperature of *110* °C is mixed with *6* kg of water (specific heat capacity *4180* J / kg / °C) at a temperature of *34* °C. Calculate the equilibrium temperature of the mixture.
   A. *39.591* °C
   B. *61.186* °C
   C. *57.587* °C
   D. *32.393* °C
   E. *35.992* °C

6. *0.55* kg of a certain substance at a temperature of *135* °C is mixed with *7.5* kg of water (specific heat capacity *4180* J / kg / °C) at a temperature of *36* °C. The equilibrium temperature of the mixture is found to be *42* °C. Calculate specific heat capacity of the substance.
   A. *4780.645* J / kg / °C
   B. *4412.903* J / kg / °C
   C. *3309.677* J / kg / °C
   D. *5148.387* J / kg / °C
   E. *3677.419* J / kg / °C

7. Calculate the heat energy required to melt *4* kg of ice (Latent heat of fusion *3.34e5* J / kg).
   A. *1603200* J
   B. *1469600* J
   C. *1870400* J
   D. *1736800* J
   E. *1336000* J

8. During vaporization, the system comprises of
   A. a mixture of liquid and gas phases.
   B. pure gas phase
   C. a mixture of solid, liquid and gas phases.
   D. a mixture of solid and gas phases.
   E. pure liquid phase.

9. Calculate the amount of heat energy required to change *0.7* kg of water at a temperature of *25* °C to steam at a temperature of *180* °C. (Specific heat capacities of ice, water and steam are *2090, 4180, 2010* J / kg / °C respectively. Latent heat of fusion and vaporization of water are *3.34e5, 2.26e6* J / kg respectively.)
   A. *2488213* J
   B. *1722609* J
   C. *1914010* J
   D. *1148406* J
   E. *2296812* J

10. Calculate the amount of heat energy required to change *0.5* kg of ice at a temperature of *-15* °C to steam at a temperature of *160* °C. (Specific heat capacities of ice, water and steam are *2090, 4180, 2010* J / kg / °C respectively. Latent heat of fusion and vaporization of water are *3.34e5, 2.26e6* J / kg respectively.)
    A. *2056567.5* J
    B. *1265580* J
    C. *1740172.5* J
    D. *949185* J
    E. *1581975* J

11.5 Ways of Transfer of Heat

There are three ways by which heat can be transferred from one point to another. These are conduction, convection and radiation.

11.5.1 Conduction

Conduction is the transfer of heat through the collision of neighboring molecules. For example, when one end of a metal is heated, the molecules at that end will vibrate vigorously. This energy is transferred to its neighboring molecules by means of collision. This process repeats itself again and again from neighbor to neighbor and eventually is transferred to the other end of the metal.

The rate of transfer of heat in a metal rod by means of conduction is proportional to the cross-sectional area (A) of the rod and to the temperature difference ($T_2 - T_1$) between the ends of the rod and inversely proportional to the length (L) of the rod.

$$Q / \Delta t = \kappa A (T_2 - T_1) / L$$

Q is amount of heat transferred from one end of the rod to the other end in a time interval Δt. That is $Q / \Delta t$ is the rate of flow of heat. The unit of measurement of rate of flow of heat (power) is J / s which is defined to be the Watt, abbreviated as W. κ is a material constant called thermal conductivity of the material. The unit of measurement for thermal conductivity is W / m / °C.

11.5.2 Convection

Convection is transfer of heat by the actual movement of molecules. This applies to liquid or gas molecules. When molecules at the bottom are heated their density decreases and they rise up. The more dense molecules at the top come down taking their place. This way heat is transferred from the bottom to the top by the actual movement of molecules. For example, when water in a dish is heated, first the molecules at the bottom of the dish are heated. Their density decreases and they rise up and the more dense molecules at the top fall down resulting in the transfer of energy from the bottom to the top.

11.5.3 Radiation

Radiation is the transfer of heat by means of electromagnetic waves. Any hot object emits electromagnetic waves. Electromagnetic waves encompasses a wide range of waves including light waves. A small subset of these waves called **infrared wave** causes sensation of heat. It is a common experience that a light bulb also causes sensation of heat in addition to sensation of vision. This is because the electromagnetic wave emitted by the light bulb contains infrared waves.

Stefan's Law states that the rate of emission of electromagnetic energy by an object is proportional to the fourth power of temperature in degree Kelvin.

$$P_e = \sigma e A T_e^{4}$$

P_e is the rate of emission of electromagnetic energy by an object of surface area A at a temperature T_e in degree Kelvin. Rate of emission of energy is equal to energy emitted per a unit time ($P_e = \Delta E / \Delta t$). The unit of measurement of emission rate is Watt. e is a constant called emissivity that depends on the properties of the surface. Its value is between *0* (for no emission) and *1* (for perfect emission). It is unitless. σ is a universal constant called Stefan-Boltzman constant. Its value is *5.7e-8* W / m^2 / °K^4.

$$\sigma = 5.7e\text{-}8 \text{ W / m}^2\text{ / °K}^4$$

An object not only emits electromagnetic energy to its environment but also absorbs electromagnetic energy from its environment. The rate of absorption of energy from the environment is proportional to the fourth power of the temperature of the environment in degree Kelvin.

$$P_a = \sigma e A T_a^{4}$$

P_a is the rate at which radiation energy is absorbed by an object of surface area A whose environment is at a temperature T_a. The net rate of emission (P_{net}) of radiation is the difference between the emission rate and absorption rate.

$$P_{net} = P_e - P_a = \sigma e A(T_e^{\,4} - T_a^{\,4})$$

11.6 Practice Quiz 11.2

Choose the best answer. Answers can be found at the back of the book.

1. Radiation is a way of transfer of heat
   A. by means of diffusion.
   B. by means of propagation of electromagnetic waves.
   C. by means of the actual movement molecules.
   D. by means of movement of air molecules.
   E. by means of collisions of neighboring molecules.

2. The rate of flow of heat in a metal rod is
   - A. is inversely proportional to the cross-sectional area of the rod
   - B. is proportional to the length of the rod.
   - C. is inversely proportional to the square of its length.
   - D. is proportional to the difference between the fourth power of the temperatures at its end points.
   - E. is proportional to the difference between the temperatures at its end points.

3. A rod made of copper (thermal conductivity *397* J / s / m / °C) has a length of *4* m and a cross-sectional radius of *0.0225* m. Its left end and right end are at temperatures of *24* °C and *125* °C respectively. Calculate the rate of transfer of heat through the rod by means of conduction.
   - A. *22.32* W
   - B. *12.754* W
   - C. *20.726* W
   - D. *14.349* W
   - E. *15.943* W

4. A rod made of copper (thermal conductivity *397* J / s / m / °C) has a length of *2.8* m and a cross-sectional radius of *0.01* m. Its left end and right end are at temperatures of *30* °C and *115* °C respectively. Calculate the amount of energy transferred from the left end to the right end in *36* minutes.
   - A. *5724.704* J
   - B. *7360.334* J
   - C. *9813.779* J
   - D. *11449.409* J
   - E. *8178.149* J

5. A spherical object of radius *0.4* m and emissivity *0.4* is at a temperature of *85* °C. Its environment is at a temperature of *25* °C. Calculate the rate at which the object is emitting radiation to its environment.
   - A. *451.802* W
   - B. *527.102* W
   - C. *602.403* W
   - D. *677.703* W
   - E. *753.003* W

6. A spherical object of radius *0.3* m and emissivity *0.9* is at a temperature of *60* °C. Its environment is at a temperature of *28* °C. Calculate the amount of radiation energy emitted by the object in *9* s.
   A. *5136.64* J
   B. *8989.121* J
   C. *6420.801* J
   D. *5778.721* J
   E. *4494.56* J

7. A spherical object of radius *0.1* m and emissivity *0.1* is at a temperature of *95* °C. Its environment is at a temperature of *34* °C. Calculate the rate at which the object is absorbing radiation from the environment.
   A. *5.09* W
   B. *6.363* W
   C. *6.999* W
   D. *8.908* W
   E. *4.454* W

8. A spherical object of radius *0.8* m and emissivity *0.7* is at a temperature of *65* °C. Its environment is at a temperature of *16* °C. Calculate the net rate of emission of radiation of the object.
   A. *1949.736* W
   B. *2144.709* W
   C. *1169.842* W
   D. *1559.789* W
   E. *2534.657* W

12 Laws of Thermodynamics

Your goals for this chapter are to learn about the three laws of thermodynamics.

There are three laws of thermodynamics. These are the zeroth law, the first law and the second law of thermodynamics. *The zeroth law of thermodynamics* states that if system A and system B are in thermodynamic equilibrium and system B and system C are in thermodynamic equilibrium, then system A and system C are also in thermodynamic equilibrium. Two systems are said to be in *thermodynamic equilibrium* if they have the same temperature.

12.1 The First Law of Thermodynamics

The *total internal energy* of a system is the sum of all the potential and kinetic energies of all the particles comprising the system. There are two ways by which the internal energy of a system can be changed. These are by the supply (loss) of heat and by doing work. Heat is energy in transit from a high temperature system to a low temperature system. Internal energy of a system increases if it gains heat and decreases if it loses heat. Heat is taken to be positive if it is supplied and negative if it is lost. Work is transfer of energy by the compression or expansion of a system. Internal energy of a system increases when the system compresses and decreases when the system expands. In other words, when a system compresses it absorbs energy from its environment and when a system expands, it loses energy to its environment. We say work is done on a system when the system compresses and work is done by the system when the system expands. Work is taken to be positive when the system expands and negative when the system compresses. Work can be expressed in terms of pressure and the change in volume brought about by the pressure.

Let's consider a piston pushing on a gas in a tube with a force F causing a change in height Δy. Then the work done by the piston is given as $W = F\Delta y$. The force F can be expresses in terms of the pressure P exerted by the piston and the cross-sectional area of the tube as $F = PA$. Therefore $W = PA\Delta y$. And $A\Delta y$ is equal to the change in volume ΔV of the gas.

$$W = P\Delta V$$

SI units Pa for pressure and m^3 for volume should be used to give Joules. If liter is used for volume then kpa should be used for pressure to get Joules.

Example: The volume of a gas decreases from *50* liters to *45* liters under the influence of a pressure of *10* kpa. Calculate the work done on the system.

Solution: $P = 10$ Pa; $V_i = 50$ L; $V_f = 45$ L; $W = ?$

$$W = P\Delta V = P(V_f - V_i) = 10 * (45 - 50) \text{ J} = -50 \text{ J}$$

A *state function* is a function whose change depends only on the initial and final states of the system irrespective of the process that brought about the change. Thus, a change in a state function can be expressed as the difference between the values of the function at the final state and the initial state of the system. If Z is an arbitrary state function, then $\Delta Z = Z_f - Z_i$. The variables that determine the state of a gas are its temperature, volume and pressure. This means, a state function is a function that depends on these variables only. The total internal energy (U) of a system is a state function. Its change is equal to the difference between its values at the final and initial states.

$$\Delta U = U_f - U_i$$

Work and heat are not state functions. They depend on the process that brought about the change and not on the initial and final state. But the difference between heat and work depends on the initial and final states of the system only. The *first law of thermodynamics* states that the difference between heat (Q) and work (W) is a state function and is equal to the change in the total internal energy of the system. The following is a mathematical statement of the first law of thermodynamics.

$$\Delta U = Q - W$$

Using the expression for work in terms of pressure and volume, the following alternative equation for the first law of thermodynamics can be obtained.

$$\Delta U = Q - P\Delta V$$

Example: A system does *50* J of energy on its environment while at the same time losing *20* J of energy to its environment. Calculate the change in its internal energy.

Solution: $Q = -20$ J; $W = 50$ J; $\Delta U = ?$

$$\Delta U = Q - W = (-20 - 50)\text{ J} = -70\text{ J}$$

Example: A system gains *40* J of heat energy while the environment is doing a work of *10*J on the system. Calculate the change in internal energy of the system.

Solution: $Q = 40$ J; $W = -10$ J; $\Delta U = ?$

$$\Delta U = Q - W = (40 - (-10))\text{ J} = 50\text{ J}$$

A *cyclic process* is a process where a system returns to its initial state. In other words, a cyclic process is a process where the initial and final states are identical. For a cyclic process $U_f = U_i$ and $\Delta U = 0$. Therefore for a cyclic process $Q = W$.

Example: A system loses *25* J of energy in a cyclic process. Calculate the work done by or on the system.

Solution: $\Delta U = 0$ (cyclic); $Q = -25$; $W = ?$

$$\Delta U = Q - W = 0$$

$$W = Q = -25\text{ J}$$

12.1.1 PV Diagrams

A PV diagram for a certain process is the graph of pressure versus volume for the process. Work can be obtained from a PV diagram as area enclosed between the Pressure versus volume curve and the volume axis. It is positive on intervals where the volume is increasing and negative on intervals where the volume is decreasing. For a cyclic process the volume will increase on one interval and decrease on the other interval. The work done will be the difference between the works on these intervals. Thus for a cyclic process the absolute value of the work done is equal to the area of the closed curve. It is positive if the process is clockwise (because the positive work is numerically greater) and negative if the process is counterclockwise (because the negative work is numerically greater).

Example: The following is a PV diagram for a certain process. Its internal energy at the states (*2* L, *2* kPa) and (*6* L, *6* kPa) are respectively *10* J and *20* J.

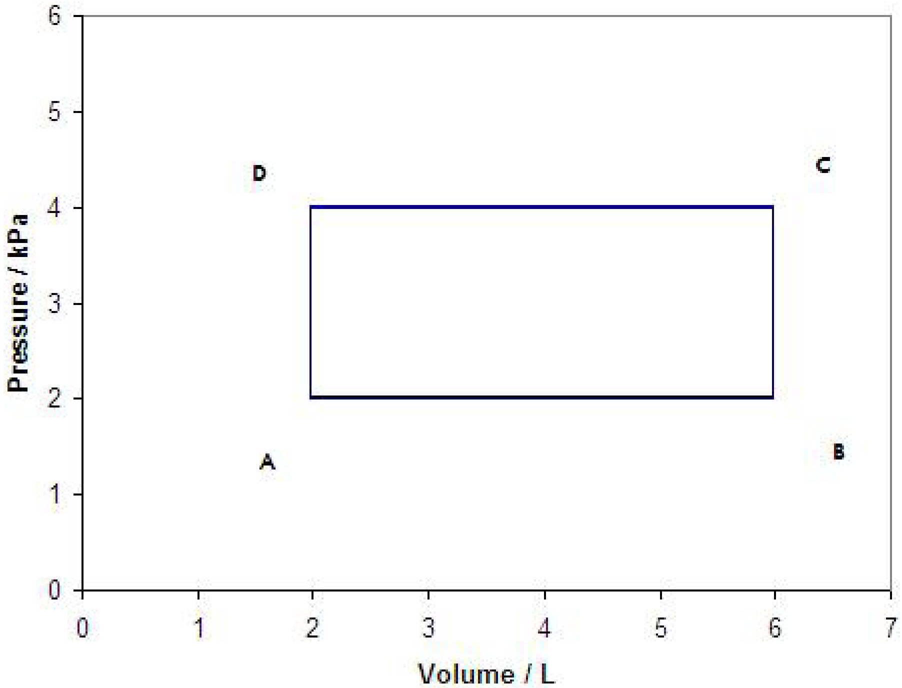


Figure 12.1

a) Calculate the work done in taking the system from the (*2* L, *2* kPa) state to the (*6* L, *6* kPa) state along the legs of the right angled triangle.

Solution: On the horizontal leg the pressure is constant at *2* kPa and the volume changes from *2* L to *6* L. On the vertical leg, the work is zero because there is no change of volume. The contribution to the work done comes only from the horizontal leg.

$P = 2$ kPa; $V_i = 2$ L; $V_f = 6$ L; $W = ?$

$$W = P\Delta V = P(V_f - V_i) = 2 * (6 - 2) \text{ J} = 8 \text{ J}$$

b) Calculate the amount of heat gained or lost when it is taken from the state (*2* L, *2* kPa) to the state (*6* L , *6* kPa) along the legs of the right angled triangle.

Solution: $U_i = 10$ J; $U_f = 20$ J; $W = 8$ J; $Q = ?$

$$\Delta U = Q - W$$

$$Q = (U_f - U_i) + W = ((20 - 10) + 8) \text{ J} = 18 \text{ J}$$

c) Calculate the work done in taking the system from the (*2* L, *2* kPa) state to the (*6* L, *6* kPa) along the straight line joining the two points (That is along the hypotenuse of the right angled triangle) and compare with the work done when the system is taken along the legs of the right angled triangle.

Solution: The pressure varies from *2* kPa to *6* kPa as the volume increases from *2* L to *6* L. The work done is equal to the area of the region enclosed by the P versus volume curve (the hypotenuse of the right angled triangle) and the V-axis. This area can be calculated by separating the region into two: The area of the right angled triangle and the area of the rectangle enclosed by the horizontal leg of the right angled triangle and the V-axis.

$$W = (4 * 4/2 + 2 * 4)\ \mathrm{J} = 16\ \mathrm{J}$$

This is different from the work done when the system is taken along the legs of the triangle which is *8* J. This indicates that work is not a state function.

d) Calculate the heat gained or lost when the system is taken from the state (*2* L, *2* kPa) to the state (*6* L, *6* kPa) along the line joining the two states (the hypotenuse) and compare with the heat obtained when the system is taken along the legs of the triangle.

Solution: $W = 16$ J; $Q = ?$

$$\Delta U = Q - W$$

$$Q = (U_f - U_i) + W = \{(20 - 10) + 16\} \text{ J} = 26 \text{ J}$$

This is different from the heat obtained when the system is taken along the legs of the right angled triangle. This indicates that heat is not a state function.

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e) Calculate the work and heat for the cyclic process that starts at the state (*2* L, *2* Pa) and ends up at (*2* L, *2* kPa) in a

i. counter clockwise direction.

Solution: For a cyclic process the work done is numerically equal to the area of the enclosed region and is negative for a counterclockwise process. Heat and work are equal for a cyclic process because the change in internal energy is zero.

$$W = Q = -(4 * 4/2) \text{ J} = -8 \text{ J}$$

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ii. clockwise direction.

Solution: The work is positive for a clockwise process.

$$W = Q = 8 \text{ J}$$

12.2 Practice Quiz 12.1

Choose the best answer. Answers can be found at the back of the book.

1. The zeroth law of thermodynamics states that
   A. The entropy of an isolated system can only increase.
   B. total internal energy is a state function.
   C. the efficiency of a heat engine can never be a 100%.

D. if system A and system B are in thermodynamic equilibrium and if system B is in thermodynamic equilibrium with system C, then systems A and C are also in thermodynamic equilibrium.

E. the total internal energy of a system is a state function and is equal to the difference between the heat supplied to (lost from) the system and the work done on (by) the system.

2. Which of the following is not a correct statement?
   A. In a PV diagram, a cyclic process is represented by a closed curve.
   B. For a cyclic process the change in internal energy is zero.
   C. For a cyclic process the total work done on (by) the system is zero.
   D. For a cyclic process the heat supplied to (lost by) a system is equal to the work done by (on) the system.
   E. A cyclic process is a process that ends at its initial state.

3. *30* J of work was done on a system of gas by the environment. If at the same time it lost *25* J of energy, calculate the change in its internal energy.
   A. *0* J
   B. *-5* J
   C. *-55* J
   D. *55* J
   E. *5* J

4. The internal energy of a system of gas changed from *600* J to *15* J while losing *650* J of heat energy. Calculate the work done by (on) the system.
   A. *-65* J
   B. *65* J
   C. *0* J
   D. *-1235* J
   E. *1235* J

5. The volume of a system of gas decreased from *85* L to *40* L under a pressure of *100* kPa while being supplied with *600* J of heat energy. Calculate the change in its internal energy.
   A. *-3900* J
   B. *5100* J
   C. *3900* J
   D. *-5100* J
   E. *0*

6. In a cyclic process, the volume of a system of gas decreased by *10* L under a pressure of *200* kPa. Which of the following is a true statement.
   A. The internal energy of the system increased by *2000* J.
   B. The system gained *2000* J of heat energy.
   C. The change in internal energy or the heat supplied (lost) cannot be determined based on the given information
   D. The change in internal energy of the system is zero.
   E. *2000* J of work was done by the system on the environment.

7. This question is based on figure 12.1. Calculate the work done in taking the system from the point D(*2* L, *4* kPa) to the point B(*6* L, *2* kPa) along path DCB.
   A. *-8* J
   B. *8* J
   C. *-16* J
   D. *24* J
   E. *16* J

8. This question is based on figure 12.1. Calculate the heat supplied (lost) in taking the system from the point A(*2* L, *2* kPa) where its internal energy is *90* J to the point C(*6* L, *4* kPa), where its internal energy is *35* J, along path ABC.
   A. *-39* J
   B. *-43* J
   C. *-47* J
   D. *-63* J
   E. *-71* J

9. This question is based on Figure 12.1. Calculate the work done by (on) the system for the cyclic process ADCBA.
   A. *-16* J
   B. *-8* J
   C. *8* J
   D. *16* J
   E. *0* J

10. This question is based on Figure 12.1. Calculate the heat supplied to (lost by) the system for the cyclic process ADCBA.
   A. *8* J
   B. *-8* J
   C. *-16* J
   D. *16* J
   E. *0* J

12.3 The Second Law of Thermodynamics

The second law of thermodynamics can be stated in terms of heat engines and entropy.

12.3.1 Heat Engines

A *heat engine* is a device used to convert heat energy into mechanical energy by taking heat from a hot reservoir and giving it off to a cold reservoir in a cyclic process. The work done by the engine is equal to the difference between the heat taken from the hot reservoir and the heat given off to the cold reservoir.

$$W = Q_H - Q_C$$

W stands for the work done by the heat engine per cycle. Q_H and Q_C stand for the heat taken from the hot reservoir per cycle and heat given off to the cold reservoir per cycle respectively. The efficiency (E) of the engine is equal to the ratio between the output energy and input energy. The input energy is equal to the heat taken off from the hot reservoir and the output energy is equal to the work done by the engine. That is $E = (Q_H - Q_C)/Q_H$

$$E = 1 - Q_C/Q_H$$

Example: A heat engine takes 80 J of energy from the hot reservoir and gives off 40 of energy to the cold reservoir per cycle.

a) Calculate the work done by the engine in one cycle.

Solution: $Q_H = 80$ J; $Q_C = 40$ J; $W = ?$

$$W = Q_H - Q_C = (80 - 40) \text{ J} = 40 \text{ J}$$

---

b) Calculate the efficiency of the engine.

Solution: $E = ?$

$$E = 1 - Q_C/Q_H = 1 - 40/80 = 0.5 = 50\%$$

12.3.2 The Carnot Engine

The *Carnot engine* is an ideal heat engine with the maximum possible efficiency for an engine that operates between given temperatures for the hot reservoir and cold reservoir. If T_H and T_C are the temperatures (in °K) of the hot and cold reservoirs respectively, then the efficiency (E_C) of the Carnot engine that operates between these temperatures (or the maximum possible efficiency for a heat engine that operates between these temperatures) is given by

$$E_C = 1 - T_C/T_H$$

If the efficiency of a Carnot engine is to be 100%, the temperature of the cold reservoir (T_C) has to be zero °K which is virtually impossible. This is essentially a statement of the second law of thermodynamics. The *second law of thermodynamics* states that it is impossible to have a heat engine with a 100% efficiency. In other words, it is impossible to have a heat engine that converts all of the heat energy obtained from the hot reservoir to mechanical energy.

Example: A heat engine takes *900* J from the hot reservoir and gives off *700* J of energy to the cold reservoir per cycle. The hot reservoir is at a temperature of *350* °C and the cold reservoir is at a temperature of *20* °C.

a) Calculate the efficiency of the engine.

Solution: $Q_C = 700$ J; $Q_H = 900$ J; $E = ?$

$$E = 1 - Q_C/Q_H = 1 - 700/900 = 22.2\%$$

---

b) Calculate the maximum possible efficiency that can be obtained from this engine.

Solution: $T_C = (20 + 273)$ °K $= 293$ °K; $T_H = (350 + 273)$ °K $= 623$ °K; $E_C = ?$

$$E_C = 1 - T_C/T_H = 1 - 293/623 = 53\%$$

12.3.2 Entropy

Entropy (S) is a state function used as a measure of the order or disorder of a system. It is defined in such a way that it increases with disorder. Since it is a state function, its change depends only on the entropies of the final and initial states of the system.

$$\Delta S = S_f - S_i$$

Entropy of a system is related with the freedom of movement of the particles comprising the system. For example the liquid state of a system has more entropy than the solid state of a system because the particles of a liquid have more freedom of movement than molecules of a solid do. Similarly, the gas state of a system has more entropy than the liquid state of a system because gas particles have more freedom of movement than particles of liquid.

Since freedom of movement of particles depends on the heat supplied to the system, entropy is related with the amount of heat supplied to a system. Change in entropy is equal to the amount of heat supplied (lost) per a unit temperature (in °K).

$$\Delta S = Q / T$$

Q is amount of heat supplied or lost at a temperature T (in °K). The unit of measurement for entropy is J / °K. In most cases the temperature would change during the process of the supply or loss of heat. In those cases methods of calculus are used to calculate the change in entropy. But during phase changes the temperature remains constant. As a result change in entropy for phase changes can be calculated easily.

Example: Calculate the change in entropy during the melting process of *0.3* kg of ice.

Solution: During the melting process, the temperature remains constant at the melting temperature of ice. The amount of heat supplied during the melting process can be obtained from $H_f = mL_f$.

$m = 0.3$ kg; $T = 0$ °C = 273 °K; $L_f = 2.26e6$ J / kg; $\Delta S = ?$

$$\Delta S = Q / T = mL_f / T = 0.3 * 2.26e6 / 273 \text{ J/°K} = 2484 \text{ J/°K}$$

The second law of thermodynamics can be stated in terms of the natural tendency of the entropy of a system. The *second law of thermodynamics* states that the entropy of an isolated system can only increase. In other words the entropy of an isolated system can go only from order to disorder. Since the universe as a whole can be treated as an isolated system, the entropy of the universe can only increase. Or, the universe can go only from order disorder.

Heat flows from high temperature system to a low temperature system because of the second law of thermodynamics. The heats gained or lost by each system are numerically equal. But the change in entropy which is heat divided by temperature is numerically smaller for the hotter object because of the division by a higher temperature. If the sum of the changes of both entropies is to be positive as called for by the second law of thermodynamics, the change in entropy of the colder system (with bigger change in entropy) should be positive. That is the colder system should gain heat and the hot system should loss heat. In other words, to be consistent with the second law of thermodynamics, heat must flow from a high temperature system to a low temperature system.

The entropy of non-isolated systems may not necessarily increase, because entropy is not the only function that determines the tendency of nature. Two state functions determine the tendency of natural systems. These are entropy and enthalpy (H). Enthalpy is essentially the energy of a system (energy at constant pressure to be exact). The natural tendency of nature is to decrease its enthalpy and to increase its entropy. An absolute tendency of nature may be obtained by combining these functions into a single function that is called Gibb's function (G) or free energy. Free energy of a system is defined to be the difference between enthalpy and the product of temperature and entropy.

$$G = H - TS$$

Nature tends to go in a direction that decreases its free energy. For example if the the free energy of the reactants of a reaction is greater than the free energy of the products, the reaction will take place spontaneously. But if the free energy of the reactants is less than the free energy of the products, the reaction cannot take place without the supply of external energy.

12.4 Practice Quiz 12.2

Choose the best answer. Answers can be found at the back of the book.

1. Which of the following statements is not a correct statement.
   A. The efficiency of a heat engine can never be a 100%.
   B. The universe tends to go from order to disorder.
   C. The entropy of the gas state is greater than the entropy of the liquid state.
   D. The entropy of an isolated system can only decrease.
   E. An isolated system can only proceed from order to disorder.

2. Entropy is a state function
   A. used as a measure of disorder (order) of a system.
   B. used as a measure of the amount of heat energy an object has.
   C. used as a measure of the amount of potential energy an object has.
   D. used as a measure of the amount of kinetic energy an object has.
   E. used as a measure of the number of particles an object has.

3. A heat engine absorbs *2000* J of energy from a hot reservoir at a temperature of *360* °C and gives off *730* J of energy to a cold reservoir at a temperature of *22* °C in a cyclic process. Calculate the amount of work done per cycle.
   A. *1397* J
   B. *508* J
   C. *1016* J
   D. *1651* J
   E. *1270* J

4. A heat engine absorbs *1900* J of energy from a hot reservoir at a temperature of *320* °C and gives off *770* J of energy to a cold reservoir at a temperature of *30* °C in a cyclic process. Calculate the efficiency of the engine.
   A. *23.789*%
   B. *47.579*%
   C. *65.421*%
   D. *77.316*%
   E. *59.474*%

5. Calculate the maximum possible efficiency for a heat engine that operates between a hot reservoir at a temperature of *320* °C and a cold reservoir at a temperature of *32* °C.
   A. *58.28*%
   B. *43.71*%
   C. *77.707*%
   D. *48.567*%
   E. *63.137*%

6. Calculate the change in entropy during the vaporization of *0.3* kg of water.(Latent heat of vaporization for water is *2.26e6* J / kg).
   A. *1999.464* J / °K
   B. *2181.233* J / °K
   C. *2363.003* J / °K
   D. *1817.694* J / °K
   E. *2544.772* J / °K

13 Vibrations and Waves

Your goals for this chapter are to learn about harmonic motion, relationship between force and harmonic motion, and waves.

A *vibration* is a back and forth motion as a function of time. Examples of vibration are motion of a pendulum and motion of an object attached to a spring. A *periodic vibration* is a vibration in which a certain pattern repeats itself again and again. The time taken for one pattern of the motion is called the *period* (T) of the motion. The unit of measurement for period is the second. The number of cycles executed per second is called the *frequency* (f) of the motion. Frequency and period are inverses of each other.

$$f = 1/T$$

The unit of measurement for frequency is $1/S$ which is defined to be the Hertz (Hz).

13.1 Harmonic Motion

A *harmonic motion* is a back and forth periodic motion that varies like a cosine or sine as a function of time. The displacement (Δx) of a harmonic motion can typically be expressed mathematically as

$$\Delta x = A \cos(\omega t + \beta)$$

A is called the amplitude of the motion and represents the maximum displacement of the motion. ω is called the angular frequency and represents the number of radians executed per second. Unit of measurement for angular frequency is rad / s. Since there are 2π radians in a cycle, angular frequency and frequency are related as follows:

$$\omega = 2\pi f$$

Example: The displacement of a certain harmonic motion varies with time according to the equation $\Delta x = 5$ cm $\cos(50t)$.

a) What is its maximum displacement?

Solution: $A = ?$

$$A = 5 \text{ cm}$$

---

b) How many oscillations does it execute per second?

Solution: $\omega = 50$ rad / s; $f = ?$

$$\omega = 2\pi f$$

$$f = \omega / 2\pi = 50 / (2\pi) \text{ Hz} = 8 \text{ Hz}$$

---

c) How long does it take to make one complete oscillation?

Solution: $T = ?$

$$T = 1 / f = 1 / 8 \text{ s} = 0.13 \text{ s}$$

---

d) Where is the particle after 10 seconds?

Solution: $t = 10$ s; $\Delta x = ?$

$$\Delta x = 5 \text{ cm} \cos(50t) = 5 \text{ cm} \cos(50*10) = -4.4 \text{ cm}$$

---

e) Show the graph of displacement versus time for this motion approximately.

Solution: To plot the graph approximately, the amplitude and the period need to be marked on a general shape of the graph of a cosine.

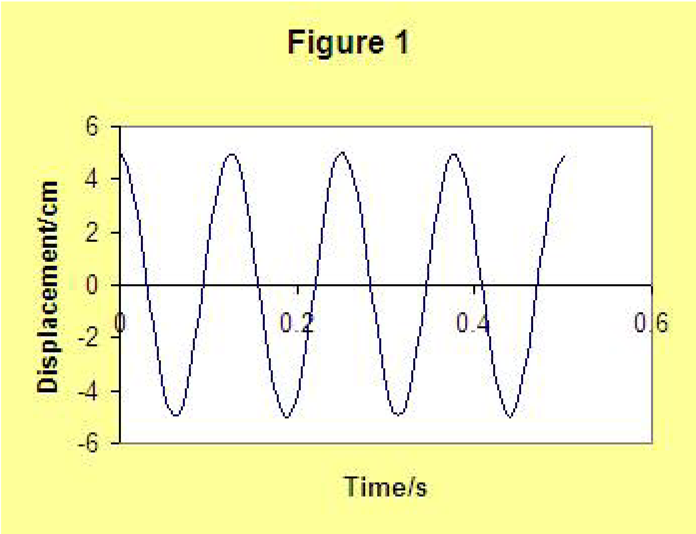

Figure 13.1

Example: The following is a graph of displacement versus time for a certain harmonic motion:

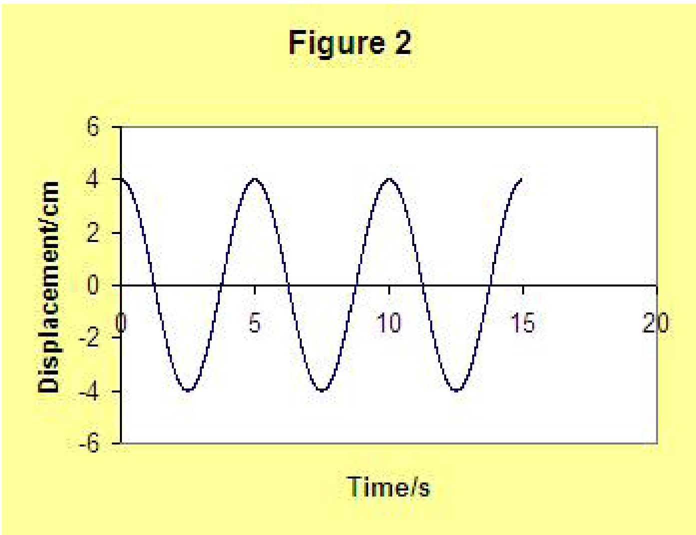

Figure 13.2

a) Determine the period of the motion.

Solution: $T = ?$

The time interval for one complete pattern is 5 s.

$$T = 5 \text{ s}$$

---

b) Determine its angular frequency.

Solution: $\omega = ?$

$$\omega = 2\pi / T = 2\pi / 5 \text{ rad/s} = 1.3 \text{ rad/s}$$

---

c) Give a mathematical formula for the displacement as function of time.

Solution: The graph is the graph of a cosine of the form $\Delta x = A \cos(\omega t)$.

$A = 4$ cm; $\omega = 1.3$ rad/s

$$\Delta x = A \cos(\omega t) = 4 \text{ cm} \cos(1.3t)$$

13.1.1 Relationship between Force and Displacement of a Harmonic Motion

For a particle executing harmonic motion, the force acting on the particle and the displacement of the particle are directly proportional and opposite in direction.

$$F = -kx$$

F stands for the force acting on the particle. x represents the displacement of the particle with respect to the relaxed position (zero-force position). k is a constant of proportionality called the force constant of the motion. Its unit of measurement is N / m. The negative sign indicates that the force and the displacement have opposite directions.

13.1.2 Relationship between Harmonic Motion and Uniform Circular Motion

A uniform circular motion is motion in a circular path with a constant speed. The force responsible for the motion is centripetal force whose direction is always towards the center of the circle. The magnitude of the centripetal force (F_c) acting on a particle of mass m revolving in a circular path of radius R with a constant speed v or constant angular speed ω is given by

$$F_c = mv^2/R = m\omega^2 R$$

The angle between the positive x-axis and the position vector of the particle at any time t is $\theta = \omega t$. Therefore the component of the centripetal force along the x-axis is given by $F_{cx} = -mw^2 R\ cos(\omega t)$. The negative sign is needed because the force and the position vector have opposite directions. The expression $R\ cos\ (\omega t)$ is equal to the x-component of the position vector of the particle which is equal to the x-coordinate, x, of the particle. Hence

$$F_{cx} = -mw^2 x$$

Since the angular speed, ω, is a constant, the expression $m\omega^2$ is a constant. It follows that the x-component of the centripetal force (F_{cx}) and the x-component of the position vector (x) are directly proportional and opposite in direction. Thus the relationship between a uniform circular motion and a harmonic motion is that the x-component of the motion (the projection of the motion on the x-axis) is a harmonic motion. From this harmonic motion, it is seen that, the force constant of a harmonic motion is related with the mass and angular speed (frequency) of the motion.

$$k = m\omega^2$$

13.2 Motion of an Object attached to a Spring: An Example of a Harmonic Motion

The motion of an object attached to a spring is governed by a law that is called Hook's law. *Hook's law* states that the force due to a spring is directly proportional and opposite in direction to the displacement (extension or compression) of the spring.

$$F_s = -\,kx$$

F_s is the force due to a spring. x is the extension or compression of the spring. k is the constant of proportionality (Force constant) called Hook's constant. Its unit is N / m. The negative sign is a representation of the fact that the spring pushes when the spring compresses and the spring pulls when the spring extends.

Example: A spring extends by *0.02* m when an object of mass *2* kg hangs from it.

a) Calculate its Hook's constant.

Solution: $x = 0.02$ m; $m = 2$ kg ($F_s = m|g|$); $k = ?$

$$|F_s| = k|x|$$

$$k = |F_s| / |x| = m|g| / |x| = 2 * 9.8 / 0.02 \text{ N/m} = 980 \text{ N/m}$$

---

b) By how much will the spring extend when subjected to a force of *12* N?

Solution: $|F_s| = 12$ N; $|x| = ?$

$$|F_s| = k|x|$$

$$|x| = |F_s| / k = 12 / 980 \text{ m} = 0.012 \text{ m}$$

Since the force acting on an object attached to a spring is proportional and opposite to the displacement of the object, the motion of an object attached to a spring is harmonic. Therefore the angular frequency of the motion is related to the Hook's constant of the spring: $k = m\omega^2$. Or

$$\omega = \sqrt{(k / m)}$$

Using the relationships $\omega = 2\pi f$ and $T = 1/f$, the frequency and the period of the motion can be expressed in terms of the mass of the object and the Hook's constant of the spring.

$$f = \{\sqrt{(k/m)}\}/(2\pi)$$

$$T = 2\pi\sqrt{(m/k)}$$

Example: An object of mass *4* kg is connected to a spring of Hook's constant *200* N/m on a frictionless horizontal surface. The spring is extended by *0.2* m and then let free to oscillate.

a) Determine its maximum displacement.

Solution: $A = ?$

Because of the principle of conservation of energy, its displacement will never exceed its initial displacement.

$$A = 0.02 \text{ m}$$

---

b) How many oscillations does it execute per second?

Solution: $m = 4$ kg; $k = 200$ N / m; $f = ?$

$$f = \{\sqrt{(k/m)}\}/(2\pi) = \{\sqrt{(200/4)}\}/(2\pi)\ \text{Hz} = 1.1\ \text{Hz}$$

---

c) How long does it take to make one complete oscillation?

Solution: $T = ?$

$$T = 1/f = 1/1.1\ \text{s} = 0.9\ \text{s}$$

---

d) Give a formula for its displacement as a function of time.

Solution: Since it started at rest from its maximum displacement, the displacement is given as $\Delta x = A\ cos\ (\omega t)$

$$\omega = 2\pi f = 2 * \pi * 1.1\ \text{rad/s} = 6.9\ \text{rad/s}$$

$$\Delta x = A\ cos\ (\omega t) = 0.02\ \text{m}\ cos\ (6.9t)$$

13.2.1 Force, Acceleration and Velocity

According to Hook's law force acting on an object attached to a spring is the negative of the product of the Hook's constant of the spring and its displacement.

$$F = -\ kx$$

According Newton's second law acceleration (a) is obtained as the ratio between force and mass.

$$a = F/m = -\ kx/m$$

An expression for the velocity can be obtained from the consideration of the principle of conservation of mechanical energy (sum of kinetic and potential energies). Since the force due to a spring is conservative, the mechanical energy of the motion of an object attached to a spring is conservative. The potential energy of a spring extended or compressed by x is given by $kx^2/2$ where k is the Hook's constant of the spring. Suppose a spring with an object of mass m attached to it is displaced by A (its amplitude) and then let free to oscillate. At this initial location it has only potential energy. Therefore its mechanical energy (ME) is equal to its initial potential energy.

$$ME = kA^2/2$$

Since its mechanical energy is conserved, its mechanical energy at an arbitrary displacement x is equal to this initial mechanical energy.

$$mv^2/2 + kx^2/2 = kA^2/2$$

Solving this equation for the speed v, the following expression for the speed at an arbitrary displacement x is obtained.

$$v = \pm \sqrt{(k/m)}\sqrt{(A^2 - x^2)}$$

Also, since $\omega = \sqrt{(k/m)}$

$$v = \pm \omega\sqrt{(A^2 - x^2)}$$

The positive sign is used when the object is moving to the right and the negative sign is used when the object is moving to the left.

Example: An object of mass 2 kg is attached to a spring of Hook's constant 150 N / m. The spring is extended by 0.03 m and then let free to oscillate. By the time it is displaced by 0.02 m

a) Calculate the force acting on it.

Solution: $k = 150$ N / m; $x = 0.02$ m; $F = ?$

$$F = -kx = -150 * 0.02 \text{ N} = -3 \text{ N}$$

---

b) Calculate its acceleration.

Solution: $m = 2$ kg; $a = ?$

$$a = F/m = -3/2 \text{ m/s}^2 = -1.5 \text{ m/s}^2$$

---

c) Calculate its speed.

Solution: $A = 0.03$ m; $v = ?$

$$\omega = \sqrt{(k/m)} = \sqrt{(150/2)} \text{ rad/s} = 8.7 \text{ rad/s}$$

$$v = \pm \omega\sqrt{(A^2 - x^2)} = \pm 8.7 * \sqrt{(0.03^2 - 0.02^2)} \text{ m/s} = \pm 0.2 \text{ m/s}$$

13.3 Practice Quiz 13.1

Choose the best answer. Answers can be found at the back of the book.

1. The unit of measurement for angular frequency is
   A. meter / second
   B. radian / second
   C. Hertz
   D. Second
   E. radian / second^2

2. Which of the following is not a correct statement.
   A. All harmonic motions are periodic.
   B. The x-component of an object undergoing a uniform circular motion is harmonic.
   C. The motion of a pendulum is generally not harmonic.
   D. The motion of an object attached to a spring is harmonic.
   E. All periodic motions are harmonic.

3. A certain harmonic motion varies with time according to the equation $x = 10$ cm $\cos(80t)$. How many cycles does it execute per second?
   A. *80* Hz
   B. *12.732* Hz
   C. *10* Hz
   D. *0.079* Hz
   E. *0.013* Hz

4. The following is the graph of displacement versus time for a certain harmonic motion.

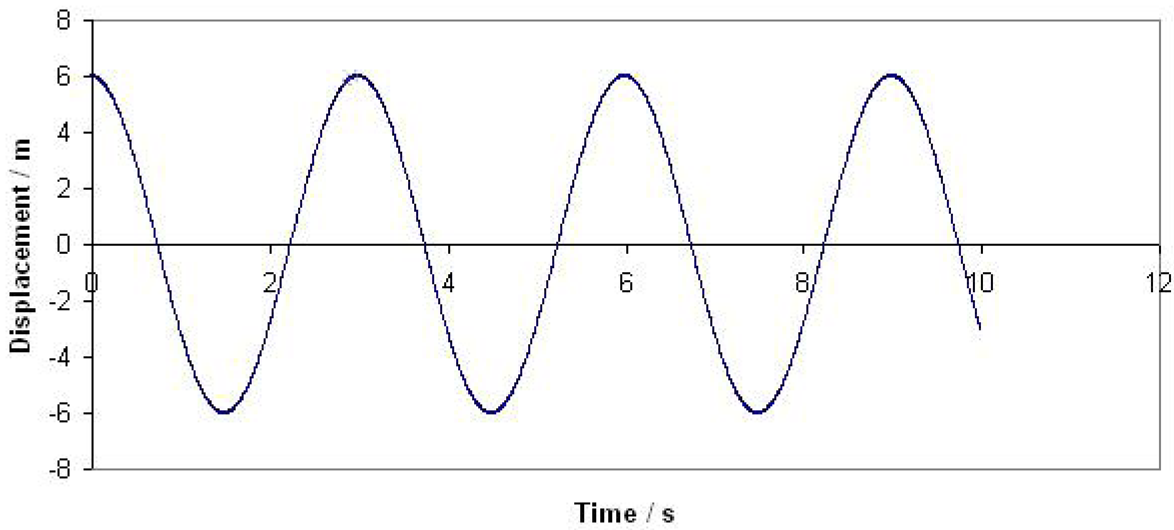

Figure 13.3

What is the frequency of the motion?
   A. *0.333* Hz
   B. *0.167* Hz
   C. *6* Hz
   D. *1.047* Hz
   E. *2.094* Hz

5. Calculate the force needed to compress a spring of Hook's constant *450* N / m by *0.15* m.
   A. *54* N
   B. *60.75* N
   C. *87.75* N
   D. *40.5* N
   E. *67.5* N

6. A spring extends by *0.2* m when an object of mass *2* kg hangs from it. By how much will it extend when an object of mass *7* kg hangs from it?
   A. *700e-3* m
   B. *420e-3* m
   C. *840e-3* m
   D. *630e-3* m
   E. *910e-3* m

7. An object of mass *2.1* kg is attached to a spring of Hook's constant *50* N / m on a friction less horizontal surface. It is extended by *0.05* m and then let free to oscillate. How many cycles does it execute per second?
   A. *0.544* Hz
   B. *0.777* Hz
   C. *0.932* Hz
   D. *1.087* Hz
   E. *0.854* Hz

8. An object is attached to a spring of Hook's constant *25* N / m on a friction less horizontal surface. It is extended by *0.05* m and then let free to oscillate. If it makes *7* cycles per second, calculate the mass of the object.
   A. *0.014* kg
   B. *0.018* kg
   C. *0.01* kg
   D. *0.009* kg
   E. *0.013* kg

9. An object of mass *0.2* kg is attached to a spring of Hook's constant *175* N / m on a friction less horizontal surface. It is extended by *0.1* m and then let free to oscillate. By the time the object passes the relaxed position of the spring, calculate the mechanical energy of the object.
   A. *700e-3* J
   B. *787.5e-3* J
   C. *612.5e-3* J
   D. *0* J
   E. *875e-3* J

13.4 Motion of a Pendulum: an Example of a Harmonic Motion

The force responsible for the motion of a pendulum is the component of gravitational force along the trajectory of the object. Let the length of the pendulum be l and the angle formed between the pendulum and the vertical be θ. The magnitude of gravitational force acting on the pendulum is equal to its weight which is $m|g|$ where m is the mass of the pendulum. The direction of this force is vertically downward. The component of this force along the circular trajectory of the pendulum (F_s) is given by

$$F_s = -\, m|g| \, sin\ (\theta)$$

The negative sign represents the fact that the component of gravity is opposite in direction to the displacement as measured from the zero-θ position. For small angles (less than *15* degrees), $sin\ (\theta)$ is approximately equal to θ. And the angle θ (in radians) is equal to the ratio between the arc length s subtended by the angle and the length of the pendulum: $\theta = s \,/\, l$. Thus, for small angles, the component of gravity along the trajectory is approximately given as follows.

$$F_s \approx -\, m|g|s \,/\, l$$

Therefore, for small angles the component of the force along the trajectory (F_s) is approximately proportional and opposite in direction to the displacement s with a constant of proportionality of $m|g|/l$. It follows that the motion of a pendulum is approximately harmonic for small angles. The constant of proportionality is the force constant of the harmonic motion and is equal to $m\omega^2$; and it follows that the angular frequency of the motion is given as

$$\omega = \sqrt{(|g|/l)}$$

Using the relationships $\omega = 2\pi f$ and $f = 1/T$, the following expressions for the frequency and period can be obtained.

$$f = \{ \sqrt{(|g|/l)}\}/(2\pi)$$

$$T = 2\pi\sqrt{(l/|g|)}$$

Example: A pendulum of length 2 m is displaced by a small angle and then let free to oscillate.

a) How long will it take to make one complete oscillation?

Solution: $l = 2$ m; $T = ?$

$$T = 2\pi\sqrt{(l/|g|)} = 2 * 3.14 * \sqrt{(2/9.8)}\ \text{s} = 2.8\ \text{s}$$

---

b) How many oscillations does it execute per second?

Solution: $f = ?$

$$f = 1/T = 1/2.8\ \text{Hz} = 0.4\ \text{Hz}$$

Example: How long does a pendulum has to be if it is to make 10 oscillations per second?

Solution: $f = 10$ Hz; $l = ?$

$$f = \{ \sqrt{(|g|/l)} \}/(2\pi)$$

Squaring both sides

$$f^2 = |g|/(4\pi^2 l)$$

$$l = |g|/(4\pi^2 f^2) = 9.8/(4 * 3.14^2 / 10^2)\ \text{m} = 0.0025\ \text{m}$$

13.5 Waves

A *wave* is a variation of a certain physical quantity as a function of position and time typically like a cosine or sine functions. An example is an ocean wave. For an ocean wave the physical quantity that varies as a function of position and time is the up and down displacement of the water molecules. If a snap shot of the ocean is taken, a wave of the vertical displacement of the water molecules as a function of position is obtained. The distance between two consecutive peaks of the wave as a function of position is called the *wavelength* (λ) of the wave. If a certain point of the ocean is considered, a wave (oscillation) of the vertical displacement of a water molecule as a function of time is obtained. The time interval between two peaks of a wave as a function of time (time taken for one oscillation) is called the **period** (T) of the wave.

The *speed of a wave* (v) is defined to be the speed of the peaks of the wave. A peak of a wave travels a distance of one wavelength in a time interval equal to the period of the wave. Thus, the speed of a wave may be obtained as a ratio between the wavelength and the period of the wave: ($v = \lambda / T$). Expressing the period in terms of frequency using the equation $T = 1/f$, the equation known as the wave equation is obtained.

$$v = \lambda f$$

Example: Consecutive peaks of an ocean wave are separated by a distance of *3* m. The water molecules take *0.4* seconds to make one complete oscillation. Calculate the speed of the wave.

Solution: $\lambda = 3$ m; $T = 0.4$ s; $v = ?$

$$f = 1/T = 1/0.4 \text{ Hz} = 2.5 \text{ Hz}$$

$$v = \lambda f = 3 * 2.5 \text{ m/s} = 7.5 \text{ m/s}$$

13.5.1 Harmonic Wave

A **Harmonic wave** is a wave that varies like a cosine or sine as a function of position and time. A harmonic wave is typically given as

$$y = A \cos(\omega t - kx)$$

A is the maximum value of the wave and is called the amplitude of the wave. ω is the angular frequency of the wave and is equal to $2\pi f$ or $2\pi / T$. k is called the wave number of the wave and is related with the wavelength as $k = 2\pi / \lambda$.

Example: A certain harmonic wave varies with position and time according to the equation $y = 0.5$ m $\cos(50t - 40x)$.

a) How long does it take to make one complete oscillation?

Solution: comparing with $y = A\cos(\omega t - kx)$:

$\omega = 50$ rad/s; $T = ?$

$$\omega = 2\pi/T$$

$$T = 2\pi/\omega = 2 * 3.14/50 \text{ rad/s} = 0.13 \text{ s}$$

---

b) Calculate the wavelength of the wave.

Solution: Comparing with $y = A\cos(\omega t - kx)$:

$k = 40$ 1/m; $\lambda = ?$

$$k = 2\pi/\lambda$$

$$\lambda = 2\pi/k = 2 * 3.14/40 \text{ m} = 0.157 \text{ m}$$

---

c. Calculate the speed of the wave.

Solution v = ?

$$v = \lambda f = \lambda / T = 0.157/0.13 \text{ m/s} = 1.2 \text{ m/s}$$

13.5.2 Wave Interference

Two (or more) waves are said to *interfere* if they meet at the same point at the same time. The net instantaneous effect is equal to the algebraic sum of the instantaneous values of the waves. That is if the two waves $y_1 = A_1 \cos(\omega t - kx_1)$ and $y_2 = A_2 \cos(\omega t - kx_2)$ interfere, then the net wave is given as $y_{net} = y_1 + y_2 = A_1 \cos(\omega t - kx_1) + A_2 \cos(\omega t - kx_2)$

Example: The following diagram is a graph of two interfering waves as a function of time. Obtain the graph of the net wave as a function of time.

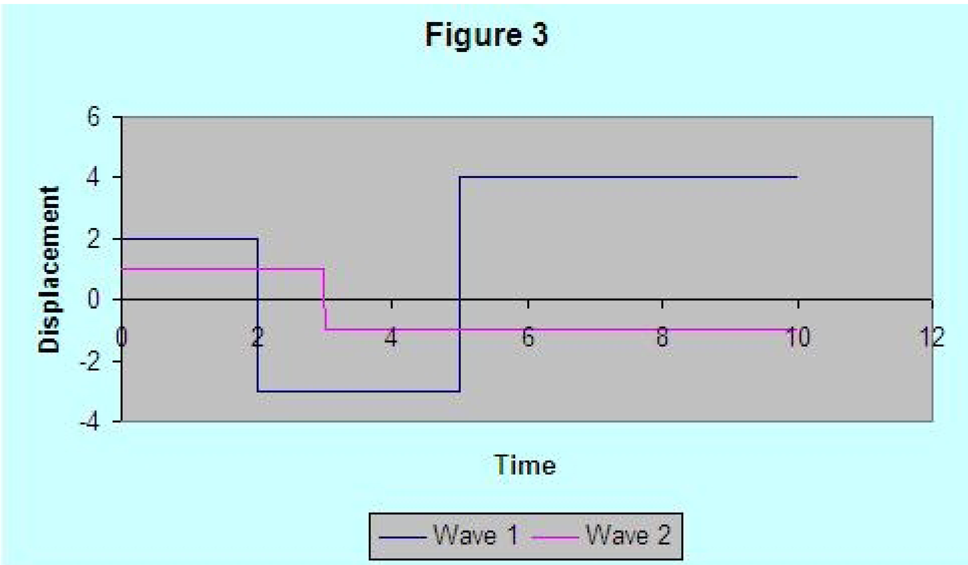

Figure 13.4

Solution: The net wave is obtained by adding the two waves instantaneously. For suitability, the graph can be divided into intervals where both waves are constant and perform the addition on the intervals separately. For example on the interval between $t = 0$ and $t = 2$ s both waves are constant. On this interval Wave 1 is equal to 2 m and Wave 2 is equal to 1 m. Therefore on this interval the net wave is equal $(1 + 2)$ m = 3 m. The graph of the net wave can be obtained by repeating this on all the intervals where both are constant. The following graph shows both waves and the net wave.

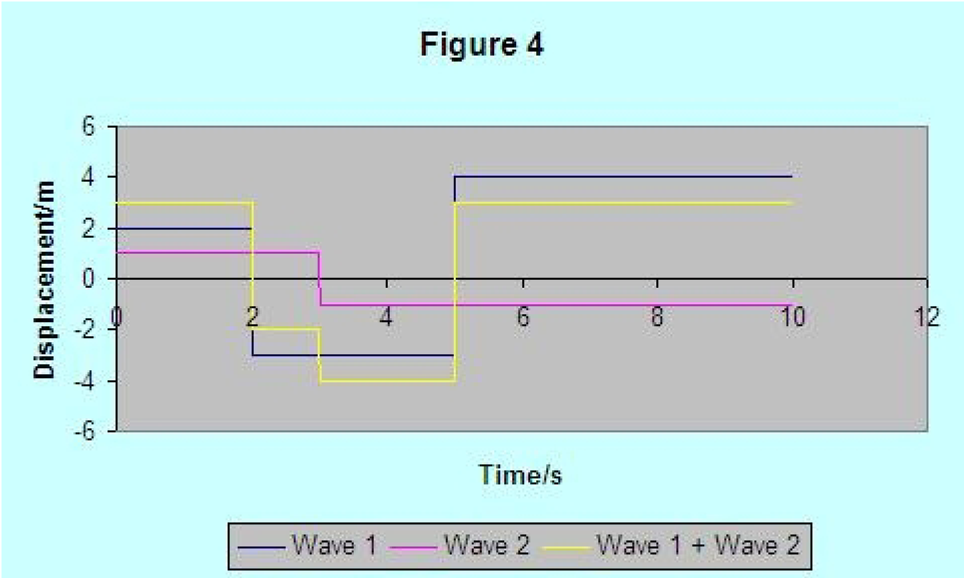

Figure 13.5

13.5.3 Constructive and Destructive Interference

Constructive interference is interference with the maximum possible effect. This happens when the two waves vibrate in phase (phase shift zero). If the waves are harmonic waves the amplitude of the net wave is the sum of the amplitudes of the interfering waves. The following diagram shows the constructive interference of two harmonic waves in phase.

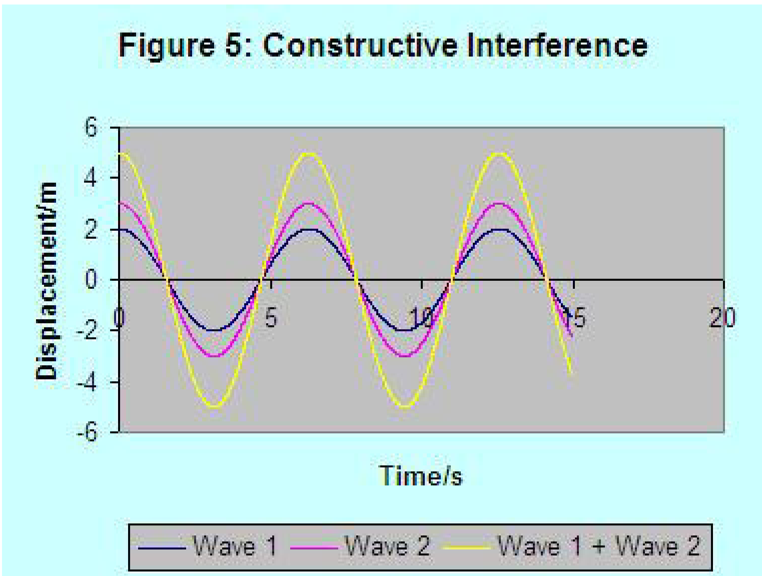

Figure 13.6

Destructive interference is interference with the minimum possible effect. It happens when the two waves vibrate out of phase (phase shift π rad). If the wave is a harmonic wave, the amplitude of the net wave is the difference between the amplitudes of the two interfering waves. The following diagram shows the destructive interference of two harmonic waves out of phase.

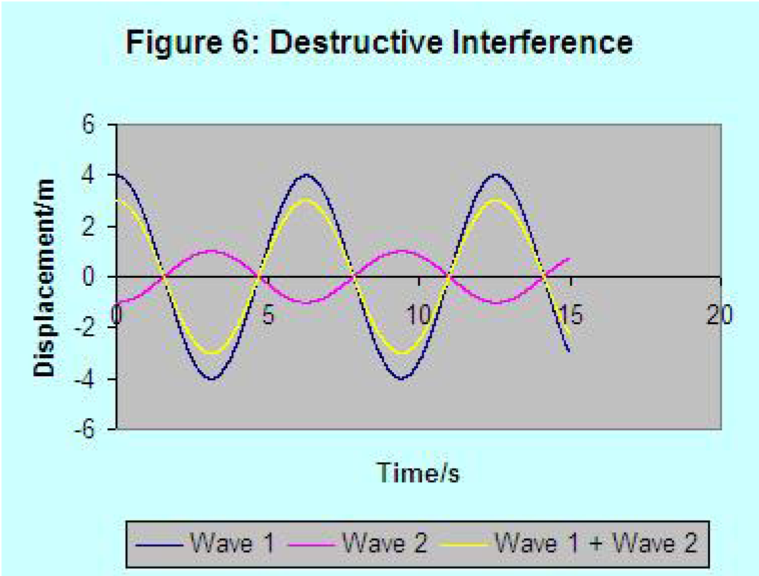

Figure 13.7

13.5.4 Reflection of Waves

Reflection of waves is the bouncing of waves from a surface. When a wave reflects from a denser medium, the reflected wave and the incident wave are out of phase; that is there is a phase shift of π radians or *180* degrees between the incident and the reflected wave. But, when a wave is reflected from a less dens medium the incident and reflected waves are in phase; that is the phase shift is zero.

13.6 Practice Quiz 13.2

Choose the best answer. Answers can be found at the back of the book.

1. The motion of a pendulum is approximately harmonic if
   A. the pendulum is displaced by a large angle (close to 90 degree)
   B. the mass of the pendulum is small.
   C. the pendulum is very short.
   D. the pendulum is very long.
   E. the pendulum is displaced by a small angle.

2. If two waves interfere with the maximum possible effect, the interference is said to be
   A. harmonic interference
   B. none of the other choices
   C. destructive interference
   D. constructive interference
   E. complementary interference

3. How long will a pendulum of length *1* m take to make one complete oscillation?
   A. *1.204* s
   B. *2.409* s
   C. *2.007* s
   D. *2.609* s
   E. *2.208* s

4. How long does a pendulum have to be if it is to make *1.5* cycles per second?
   A. *0.132* m
   B. *0.11* m
   C. *0.121* m
   D. *0.154* m
   E. *0.077* m

5. By what factor would the period of a pendulum change when its length is multiplied by a factor of *3*?
   A. *3*
   B. *1.732*
   C. *9*
   D. *0.577*
   E. *0.111*

6. A pendulum takes *1.75* seconds to make one complete oscillation on earth. How long will it take to make one complete oscillation on the moon where gravitational acceleration is $(1/6)^{th}$ of that on earth?
   A. *4.715* s
   B. *3.429* s
   C. *4.287* s
   D. *6.001* s
   E. *5.573* s

7. The peaks of an ocean are approaching the beach with a speed of *3* m/s. If the water molecules take *2* s to make one complete cycle, calculate the distance between two consecutive peaks.
   A. *8.4* m
   B. *6* m
   C. *7.8* m
   D. *7.2* m
   E. *6.6* m

8. A certain harmonic motion varies with time and position according to the equation $y = 0.4$ m $\cos(30t - 5x)$. Calculate the speed of the wave.
   A. *7.2* m/s
   B. *7.8* m/s
   C. *5.4* m/s
   D. *6* m/s
   E. *8.4* m/s

9. This question is based on Figure 13.4. What is the value of the net wave during the first *2* seconds?
   A. *1* m
   B. *2* m
   C. *-2* m
   D. *4* m
   E. *3* m

10. If the harmonic waves $y_1 = 14\text{ m}\cos(20t - 40x)$ and $y_2 = 3\text{ m}\cos(20t - 80x)$ interfere destructively, then the amplitude of the net wave is
    A. *40* m
    B. *60* m
    C. *11* m
    D. *17* m
    E. *100* m

14 Sound Waves

Your goals for this chapter are to learn about nature of sound waves, intensity of sound waves, speed of sound waves, and standing waves.

Sound waves are produced by vibrating objects. As an object vibrates back and forth the pressure (density) of the molecules in its vicinity alternates between a maximum (compression) and a minimum (rarefaction). This variation of pressure is propagated through air as sound wave. For sound waves, the physical quantity that` varies as a function position and time is the pressure of the molecules. The wavelength of sound is the distance between two consecutive compressions. A given molecule vibrates back and forth in the direction of the propagation of sound. The period of the sound is equal to the time taken for one complete back and forth oscillation and is equal to the period of the vibrating object.

Sound wave is a longitudinal wave. *Longitudinal Wave* is a wave where the direction of the vibration of molecules is the same as the direction of propagation of energy. A wave in which the direction of vibration of the molecules is perpendicular to the direction of propagation of energy is called a *transverse wave*. An example of transverse wave is an ocean wave. The molecules vibrate in a direction perpendicular to the surface of the ocean while the wave travels parallel to the surface of the ocean.

Sound waves are classified into three based on frequency. Sound waves that can be detected by the human ear are called *audible*. This includes sound waves whose frequency is between *20* and *20, 000* Hz. Sound waves whose frequency is below the audible range are called *infrasonic*. An example is a wave due to earthquake. It can be felt but it can't be heard because its frequency is too low. Sound waves whose frequencies are above the audible range are called *supersonic*. These waves can be detected by dogs.

Human sensation of sound is related to the properties of sound waves. The loudness of sound is related with the amplitude of sound wave. The greater the amplitude, the greater the loudness. The pitch of sound is related with the frequency of sound wave. The greater the frequency the greater the pitch of the sound.

14.1 Relationship between Speed of Sound and Temperature

Speed of sound in air at a temperature of $0\ ^\circ C$ is 331 m / s. Speed of sound (v) is proportional to the square root of temperature in °K (T). Therefore the ratio between speed of sound and square root of temperature is a constant: $v_1 / \sqrt{T_1} = v_2 / \sqrt{T_2}$. Using the fact that the speed of sound at zero °C is 331 m / s ($v_2 = 331$ m / s and $T_2 = 273$ °K) the following relationship between speed of sound and temperature can be obtained (subscripts not needed anymore).

$$v = 331 \text{ m/s} \sqrt{ (T / {}^\circ\text{K}) / 273 }$$

The temperature (T) in this equation should be in °K. If the temperature is given in °C, first it has to be converted to °K before using this equation. This equation can also be expressed in terms of temperature in °C by using the relationship between °K and °C: $T/°\text{K} = T/°\text{C} + 273$.

$$v = 331 \text{ m/s} \sqrt{1 + (T/°\text{C})/273}$$

Example: Calculate the speed of sound at a temperature of 25 °C.

Solution: $T = 25$ °C; $v = ?$

$$v = 331 \text{ m/s} \sqrt{1 + (T/°\text{C})/273} = 331 \text{ m/s} \sqrt{1 + (25/273)} = 345.8 \text{ m/s}$$

Example: Calculate the wavelength of the sound produced by a 500 Hz tuning fork when the temperature is 20 °C.

Solution: $f = 500$ Hz; $T = 20$ °C ($v = 331 \text{ m/s} \sqrt{1 + (T/°\text{C})/273}$); $\lambda = ?$

$$v = 331 \text{ m/s} \sqrt{1 + (T/°\text{C})/273} = 331 \text{ m/s} \sqrt{1 + (20/273)} = 342.9 \text{ m/s}$$

$$v = \lambda f$$

$$\lambda = v/f = 342/500 \text{ m} = 0.7 \text{ m}$$

14.2 Intensity of Sound Waves

Intensity (I) of sound waves is defined to be sound energy (E) per a unit perpendicular area ($A_{\perp}$) per a unit time.

$$I = (E/t)/A_{\perp}$$

Energy per a unit time is defined to be power (P). The unit of power is J/s which is defined to be the Watt, abbreviated as W.

$$P = E/t$$

Alternatively, intensity may be defined as power per a unit perpendicular area.

$$I = P/A_{\perp}$$

14.2.1 Intensity of a Spherical Sound Wave

A *spherical sound wave* is a sound wave that propagates from a certain source in all directions uniformly. The intensity of a spherical wave is a constant on spherical surfaces centered at the source. The intensity due to a spherical wave at a distance r from the source may be obtained as a ratio between the power (P) of the source and the surface area of the spherical surface of radius r (because the spherical surface is perpendicular to the direction of propagation of sound energy which is radial). The area of a sphere of radius r is equal to $4\pi r^2$.

$$I = P/(4\pi r^2)$$

For a source of a given power, the intensity of a spherical wave is inversely proportional to to the square of the radius which means the product of the intensity and the square of the radius is a constant for any radius.

$$I_1 r_1^2 = I_2 r_2^2$$

Example: For a spherical sound wave produced by a *200* W speaker, calculate the intensity of the sound at a distance of *5* m from the speaker.

Solution: $P = 200$ W; $r = 5$ m; $I = ?$

$$I = P/(4\pi r^2) = 200/(4 * 3.14 * 5^2) \text{ W} = 0.64 \text{ W}$$

Example: The intensity due to a certain spherical wave at a distance of *4* m is *0.4* W/m^2. Calculate the intensity at a distance of *12* m.

Solution: $r_1 = 4$ m; $I_1 = 0.4$ W/m^2; $r_2 = 12$ m; $I_2 = ?$

$$I_1 r_1^2 = I_2 r_2^2$$

$$I_2 = I_1 r_1^2/r_2^2 = 0.4 * 4^2/12^2 = 0.044 \text{ W/m}^2$$

14.2.2 Relationship between Intensity and Loudness of Sound

Loudness of sound increases logarithmically with the increase of intensity of sound. The minimum intensity of sound that can be detected by the human ear is 10^{-12} W/m^2. And the maximum intensity that can be tolerated by the human ear is *1* W/m^2. The unit of measurement for loudness is decibel abbreviated as dB. Loudness (β) in decibels is mathematically related with the intensity of sound (I) as follows:

$$\beta = 10 \log (I/I_o)$$

Where $I_o = 10^{-12}$ W/m^2 is the minimum possible intensity that can be tolerated by the human ear. The following are some logarithmic identities that may be useful in using this equation.

$$y = \log (x) \text{ if } x = 10^y$$

$$\log (xy) = \log (x) + \log (y)$$

$$\log (x/y) = \log (x) - \log (y)$$

$$\log (10^x) = x$$

$$\log (10) = 1$$

$$\log (1) = 0$$

Example: Calculate the loudness level in decibels of a sound at a distance of *20* m from a *500* W speaker. Assume the sound waves to be spherical.

Solution: $P = 500$ W; $r = 20$ m $(I = P/(4\pi r^2)$; $\beta = ?$

$$I = P/(4\pi r^2) = 500/(4 * 3.14 * 20^2) = 0.1 \text{ W/m}^2$$

$$\beta = \log (I/I_o) = 10 * \log (0.1/10^{-12}) \text{ dB} = 110 \text{ dB}$$

14.3 Speed of Sound in Fluids and Metals

14.3.1 Speed of Sound in Fluids

The speed of sound in a fluid is directly proportional to the square root of the bulk modulus (B) of the fluid and inversely proportional to the square root of the density (ρ) of the fluid.

$$v = \sqrt{(B/\rho)}$$

14.3.2 Speed of Sound in Metals

The Speed of sound in metals is directly proportional to the square root of the Young's modulus (Y) of the metal and inversely proportional to the square root of the density (ρ) of the metal.

$$v = \sqrt{(Y/\rho)}$$

14.4 Doppler's Effect

Doppler's Effect refers to the change in the frequency (pitch) of the sound heard by an observer due to the relative speed between source and observer. It is a common experience that when a car approaches an observer while blowing its horn, the pitch of the sound increases and as the car passes away the pitch of the sound decreases.

14.4.1 Case 1: Source Stationary and Observer Moving

Suppose a stationary source is producing sound of frequency f, an observer is moving with a speed v_o and the speed of sound is v, then the modified frequency $f\,'$ heard by the observer is given by

$$f\,' = f\,\{ (v \pm v_0)/v \}$$

The plus sign is used when the observer moves towards the source and the minus sign when the observer moves away from the source.

14.4.2 Case 2: Observer Stationary and Source Moving

Suppose the source is moving with a speed v_s, the speed of sound is v and the source is producing sound of frequency f. Then the modified frequency $f\,'$ heard by the observer is given by

$$f\,' = f\,\{\,(v\,/\,(v \pm v_s)\,\}$$

The plus sign is used when the source moves away from the source and the minus sign when the source moves towards the observer.

Example: A stationary car is producing sound of frequency *800* Hz when the temperature is *23* °C. Calculate the frequency of the sound heard by an observer, when the observer approaches the car with a speed of *30* m / s.

Solution: $f = 800$ Hz; $T = 23$ °C ($v = 331$ m / s $\sqrt{\{\,1 + (T/°\mathrm{C})/273\,\}}$); $v_o = 30$ m / s; $f\,' = ?$

$$v = 331\ \mathrm{m/s}\ \sqrt{\{\,1 + (T/°\mathrm{C})/273\,\}} = 331 * \sqrt{\{\,1 + (23/273)\,\}}\ \mathrm{m/s} = 344.7\ \mathrm{m/s}$$

$$f\,' = f\,\{\,(v + v_o)/v\,\} = 800 * \{\,(344.7 + 30)/344.7\,\}\ \mathrm{m/s} = 869.6\ \mathrm{m/s}$$

Example: A car, producing sound of frequency *600* Hz, is going away from a stationary observer with a speed of *10* m / s. Calculate the frequency of the sound heard by the observer. The temperature is *25* °C.

Solution: $v_s = 10$ m / s; $f = 600$ Hz; $T = 25$ °C ($v = 331$ m / s √{ 1 + (T / °C) / 273 }); $f ' = ?$

$$v = 331 \text{ m/s} \sqrt{1 + (T/°\text{C})/273} = 331 * \sqrt{1 + (25/273)} \text{ m/s} = 345.8 \text{ m/s}$$

$$f ' = f \{ v/(v + v_s) \} = 600 * \{ 345.8/(345.8 + 10) \} \text{ Hz} = 583.1 \text{ Hz}$$

14.5 Practice Quiz 14.1

Choose the best answer. Answers can be found at the back of the book.

1. What is the unit of measurement for intensity of sound?
   - A. Watt / meter
   - B. Watt / meter ^{2}
   - C. Watt
   - D. Joule / meter ^{2}
   - E. Joule / meter

2. Speed of sound
   - A. is proportional to square root of temperature
   - B. is proportional to the square of temperature.
   - C. is proportional to temperature
   - D. is inversely proportional to temperature
   - E. is inversely proportional to the square of temperature.

3. Calculate the speed of sound in air at a temperature of *50* ˚C.
   - A. *216.023* m / s
   - B. *324.034* m / s
   - C. *360.038* m / s
   - D. *288.03* m / s
   - E. *252.026* m / s

4. Calculate the frequency of sound of wavelength *4* m at a temperature of *45* °C.
   - A. *62.517* Hz
   - B. *53.586* Hz
   - C. *89.31* Hz
   - D. *71.448* Hz
   - E. *98.241* Hz

5. A loud speaker is producing spherical sound waves. Calculate the power of the loud speaker if the intensity of the sound at a distance of *9* m from the loud speaker is *1* $\mathrm{W/m^2}$.
   A. *712.513* W
   B. *1221.451* W
   C. *1425.026* W
   D. *1119.664* W
   E. *1017.876* W

6. A loud speaker is producing spherical sound waves. If the intensity of the sound at a distance of *14* m is *0.04* $\mathrm{W/m^2}$, at what distance would the intensity be *0.15* $\mathrm{W/m^2}$.
   A. *10.121* m
   B. *5.061* m
   C. *7.23* m
   D. *7.953* m
   E. *8.675* m

7. A loud speaker of power *75* W is producing spherical sound waves. Calculate the loudness level of the sound at a distance of *14* m from the loud speaker.
   A. *104.836* dB
   B. *125.803* dB
   C. *73.385* dB
   D. *62.902* dB
   E. *94.352* dB

8. Calculate the speed of sound in water. (Density and bulk modulus of water are respectively *1000* $\mathrm{kg/m^3}$ and *2.1e9* Pa.)
   A. *1449.138* m/s
   B. *1304.224* m/s
   C. *1159.31* m/s
   D. *1014.396* m/s
   E. *2028.793* m/s

9. A man on a bicycle is approaching a stationary car that is producing sound of frequency *1100* Hz with a speed of *7* m/s at a day when the temperature is *30* °C. Calculate the frequency of the sound heard by the man.
   A. *1458.706* Hz
   B. *673.249* Hz
   C. *1009.873* Hz
   D. *897.665* Hz
   E. *1122.081* Hz

10. A car that is producing sound of frequency *1300* Hz is approaching a stationary man with a speed of *10* m/s at a day when the temperature is *30* °C. Calculate the frequency of the sound heard by the man.
    A. *936.866* Hz
    B. *1338.381* Hz
    C. *1472.219* Hz
    D. *1070.704* Hz
    E. *1873.733* Hz

14.6 Speed of a Wave in a string

When one end of a string is disturbed, the disturbance will travel along the string as a wave. The speed of this wave (v) is directly proportional to the square root of the tension (T) in the string and is inversely proportional to the square root of the mass per unit length (μ) of the string.

$$v = \sqrt{(T/\mu)}$$

For a uniform string, the mass per unit length of the string can be obtained as the ratio between the mass (m) and the length (l) of the string: $\mu = m/l$.

Example: The tension in a string of mass *0.008* kg and length *2* m is *300* N. Calculate the speed of a wave in the string.

Solution: $T = 300$ N; $m = 0.008$ kg; $l = 2$ m ($\mu = m/l$); $v = ?$

$$\mu = m/l = 0.008/2 \text{ kg/m} = 0.004 \text{ kg/m}$$

$$v = \sqrt{(T/\mu)} = \sqrt{(300/0.004)} \text{ m/s} = 273.9 \text{ m/s}$$

14.7 Standing Waves

A *standing wave* is a wave with equally spaced points of zero vibration. The following diagram shows a standing wave formed in a string clamped at both ends.

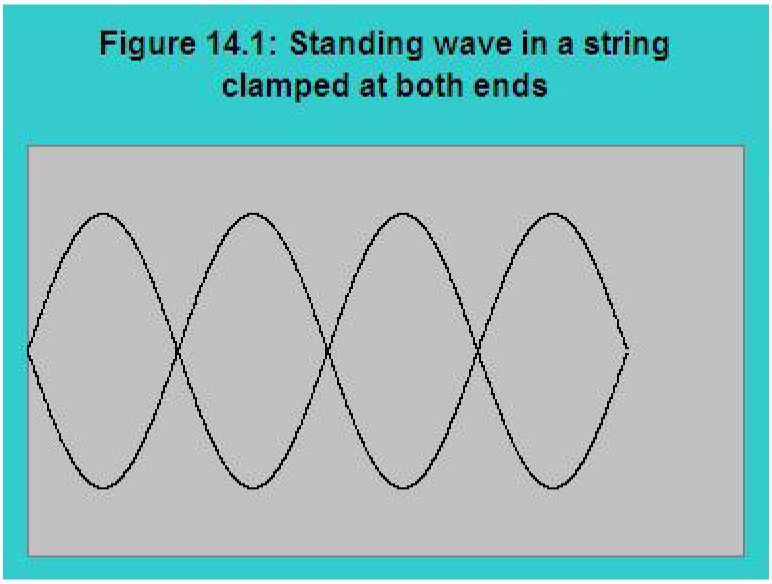

Figure 14.1

The points of zero vibration (such as the clamped ends of the string are called *nodes* and the points of maximum vibration are called *antinodes*. A part of the wave between two consecutive nodes is called a *loop*.

A standing wave is formed when an oncoming and a reflected wave interfere. For example, when one of the ends of a string clamped at both ends is being disturbed, the disturbance will travel to the other end as a wave and reflected with a phase shift of π. The oncoming wave and the reflected wave will interfere. For certain lengths of the string, the oncoming and reflected waves interfere to form a standing wave.

The length of one loop (l_{loop}) of a standing wave is equal to half of the wave length of the wave.

$$l_{loop} = \lambda/2$$

If n is the number of loops in a standing wave, then the length of the standing wave (L) may be obtained by multiplying the number of loops by the length of one loop.

$$L = nl_{loop} = n\lambda / 2$$

An expression for the frequency of a standing wave in terms of the number of loops can be obtained using the wave equation ($v = \lambda f$).

$$f = v / (2L / n) = nv / 2L$$

Example: A standing wave formed in a string of length 5 m has 6 loops. Calculate the wavelength of the wave.

Solution: $L = 5$ m; $n = 6$; $\lambda = ?$

$$L = n\lambda / 2$$

$$\lambda = 2L / n = 2 * 5 / 6 \text{ m} = 1.7 \text{ m}$$

14.7.1 Harmonics of a standing Wave

For a given length, only certain wavelengths of the wave can form standing waves. The waves with these wavelengths are called the *harmonics* of the standing wave. The harmonic with the largest wavelength or smallest frequency is called the fundamental harmonic or the first harmonic. The subsequent harmonics are identified as 2^{nd}, 3^{rd}, …, n^{th} harmonics.

14.7.2 Harmonics of a Standing Wave of a String Clamped at both Ends

Both ends must be nodes. The following diagram shows the first three harmonics of a standing wave in a string clamped at both ends.

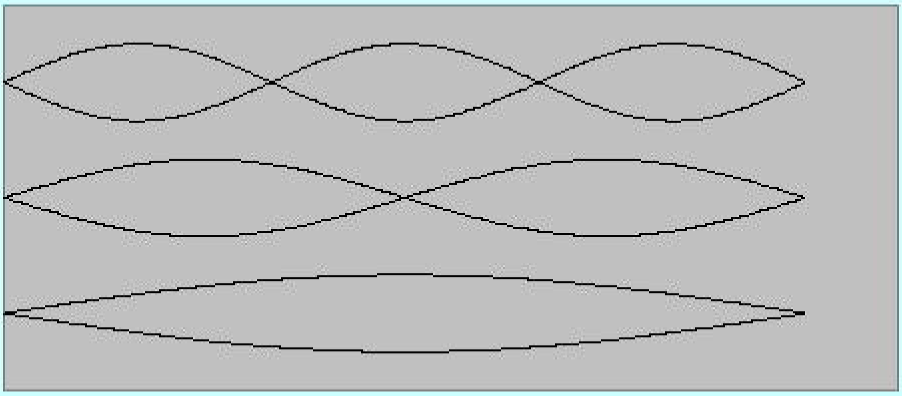

Figure 14.2: The first, second and third (from bottom to top) harmonics of the standing wave of a string clamped at both ends.

Figure 14.2

The first harmonic has one loop. Since one loop equals half of the wavelength, the wavelength (λ_1) and frequency (f_1) of the first harmonic are given as $\lambda_1 = 2L$ and $f_1 = v/(2L)$.

The second harmonic has two loops. Since one loop is half of the wavelength, the wavelength of the second harmonic is given by $\lambda_2 = L$ and the frequency of the second harmonic is given by $f_2 = 2(v/(2L)) = 2f_1$.

The n^{th} harmonic has n loops. Therefore the wavelength of the n^{th} harmonic is given by

$$\lambda_n = 2L/n$$

The frequency of the n^{th} harmonic can be obtained by dividing the speed of the wave by the wavelength of the n^{th} harmonic.

$$f_n = n(v/(2L)) = n f_1$$

The frequency of the n^{th} harmonic is n times the frequency of the first harmonic.

Example: A string clamped at both ends has a length of *4* m. It has a mass of *0.006* kg. There is a tension of *500* N in the string.

a) Calculate the speed of the wave in the string.

Solution: $L = 4$ m; $m = 0.006$ kg ($\mu = m/L$); $T = 500$ N; $v = ?$

$$\mu = m/L = 0.006/4 \text{ kg/m} = 0.0015 \text{ kg/m}$$

$$v = \sqrt{(T/\mu)} = \sqrt{(500/0.0015)} \text{ m/s} = 577.4 \text{ m/s}$$

---

b) Calculate the wavelength and the frequency of the fundamental (first) harmonic.

Solution: $\lambda_1 = ?$; $f_1 = ?$

$$\lambda_1 = 2L = 2 * 4 \text{ m} = 8 \text{ m}$$

$$f_1 = v/\lambda_1 = 577.4/8 \text{ Hz} = 72.2 \text{ Hz}$$

Calculate the frequency and the wavelength of the seventh harmonic.

Solution: $n = 7$; $f_7 = ?$; $\lambda_7 = ?$

$$f_7 = nf_1$$

$$f_7 = 7f_1 = 7 * 72.2 \text{ Hz} = 505.4 \text{ Hz}$$

$$\lambda_7 = v/f_7 = 544.7/505.4 \text{ m} = 1.1 \text{ m}$$

14.7.3 Harmonics of a sound resonance in a tube open at both ends

The standing wave should have antinodes at both ends of the pipe, because the molecules are free to vibrate over there. The following diagram shows the first, second and third harmonics of this standing wave.

Figure 14.4: The first, second and third (from bottom to top) harmonics of a sound resonance in a tube open at both ends.

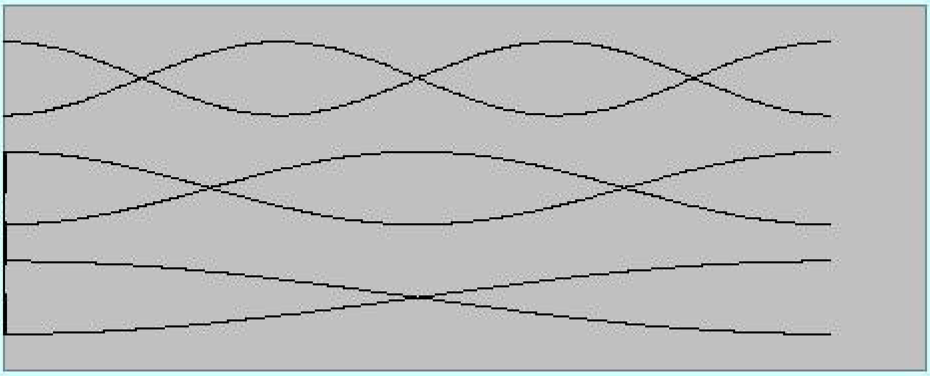

Figure 14.3

The first harmonic contains one loop. Thus $L = (\lambda_1 / 2)$ and the wavelength of the first harmonic is given by $\lambda_1 = 2L$. The frequency of the first harmonic is obtained by dividing the speed of the wave by the wavelength of the first harmonic:$f_1 = v / 2L$. The second harmonic contains two loops. Therefore $L = 2(\lambda_2 / 2)$ and the wavelength and frequency of the second harmonic are given as $\lambda_2 = L$ and $f_2 = v / L = 2(v / (2L)) = 2f_1$ respectively.

The n^{th} harmonic has n loops. Hence $L = n(\lambda_n/2)$ and the wavelength and frequency of the n^{th} harmonic are given by $\lambda_n = 2L / n$ and $f_n = nv / (2L) = n(v / (2L))$ respectively. The frequency of the n^{th} harmonic may be expressed in terms of the frequency of the first harmonic by making use of the equation $f_1 = v/2L$

$$f_n = n\, f_1$$

The frequency of the n^{th} harmonic is n times of the frequency of the first harmonic.

Example: A sound resonance is formed on a *2* m tube open at both ends. The temperature is *20* °C.

a) Calculate the wavelength and the frequency of the first harmonic.

Solution: $L = 2$ m; $T = 20$ °C ($v = 331$ m / s $\sqrt{1 + (T/°\mathrm{C})/273}$)

$$v = 331 \text{ m/s} \sqrt{1 + (T/°\mathrm{C})/273} = 331 * \sqrt{1 + (20/273)} \text{ m/s} = 342.9 \text{ m/s}$$

$$\lambda_1 = 2L = 2 * 2 \text{ m} = 4 \text{ m}$$

$$f_1 = v/\lambda_1 = 342.9/4 \text{ Hz} = 85.7 \text{ Hz}$$

---

b) Calculate the frequency and wavelength of the ninth harmonic.

Solution: $n = 9$; $f_9 = ?$; $\lambda_9 = ?$

$$f_n = nf_1$$

$$f_9 = 9f_1 = 9 * 85.7 \text{ Hz} = 771.3 \text{ Hz}$$

$$\lambda_9 = v/f_9 = 342.9/771.3 \text{ m} = 0.44 \text{ m}$$

14.7.4 Harmonics of Sound Resonance in a Pipe Open at One End Closed at the Other

The standing wave should have an antinode on the open end and a node on the closed end. The following diagram shows the first three harmonics.

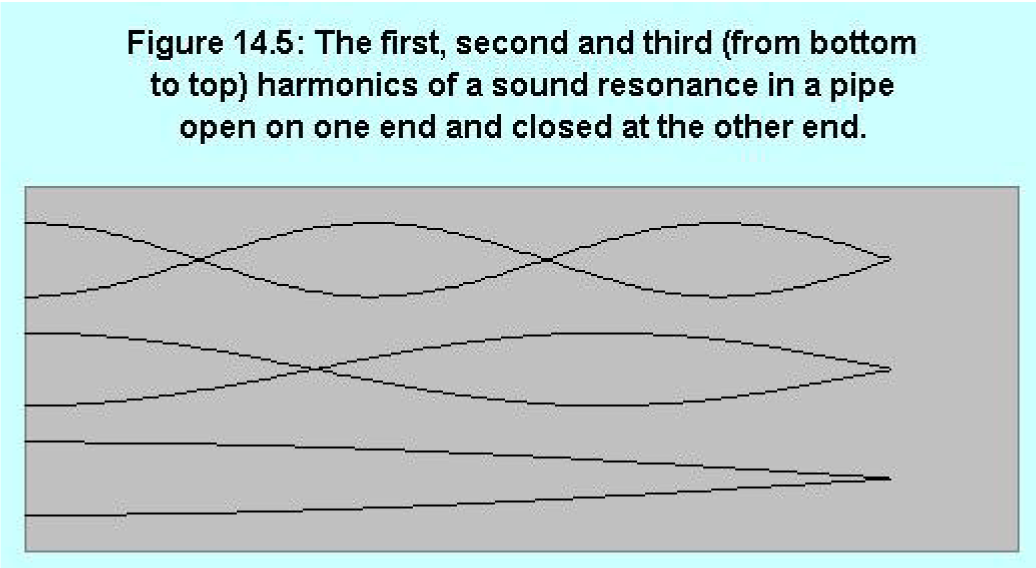

Figure 14.4

The first harmonic one is half of a loop long. Therefore $L = (1/2)(\lambda_1/2)$ and the wavelength and frequency of the first harmonic are given as $\lambda_1 = 4L$ and $f_1 = v/4L$ respectively. The second harmonic contains one and half loops and $L = (3/2)(\lambda_2/2)$. Thus the wavelength and frequency of the second harmonic are given by $\lambda_2 = 4L/3$ and $f_2 = 3v/4L = 3(v/(4L)) = 3f_1$.

The n^{th} harmonic has $(2n - 1)/2$ loops. Thus $L = \{ (2n - 1)/2 \}(\lambda_n/2)$ and the wavelength and frequency of the n^{th} harmonic are given as $\lambda_n = 4L/(2n - 1)$ and $f_n = (2n - 1)v/4L$. f_n can be expressed in terms of f_1 by making use of the equation $f_1 = v/(4L)$.

$$f_n = (2n - 1)\, f_1$$

The frequency of the n^{th} harmonic is $(2n - 1)$ times the frequency of the first harmonic.

Example: Sound resonance is formed on a *1* m tube open at one end closed at the other when the temperature is *20* °C.

a) Calculate the wavelength and frequency of the first harmonic.

Solution: $L = 2$ m; $T = 20$ °C; $\lambda_1 = ?$; $f_1 = ?$

$$\lambda_1 = 4L = 8 \text{ m}$$

$$f_1 = v/\lambda_1$$

$$v = 331 \text{ m/s} \sqrt{\{1 + (T/°\text{C})/273\}} = 331 * \sqrt{(1 + 20/273)} \text{ m/s} = 342.9 \text{ m/s}$$

$$f_1 = 342.9/8 \text{ Hz} = 42.9 \text{ Hz}$$

b) Calculate the wavelength and frequency of the fifth harmonic.

Solution: $n = 5$; $\lambda_5 = ?$; $f_5 = ?$

$$f_n = (2n - 1)\, f_1$$

$$f_5 = (2 * 5 - 1) * 42.9 \text{ Hz} = 386.1 \text{ Hz}$$

$$\lambda_5 = v / f_5 = 342.9 / 386.1 \text{ m} = 0.89 \text{ m}$$

14.8 A Beat

A *beat* is a sound wave with points of zero-vibration (no sound) separated by equal intervals of time. A beat is formed when two waves with close frequencies interfere. The following diagram shows a beat formed when two waves with close frequencies interfere.

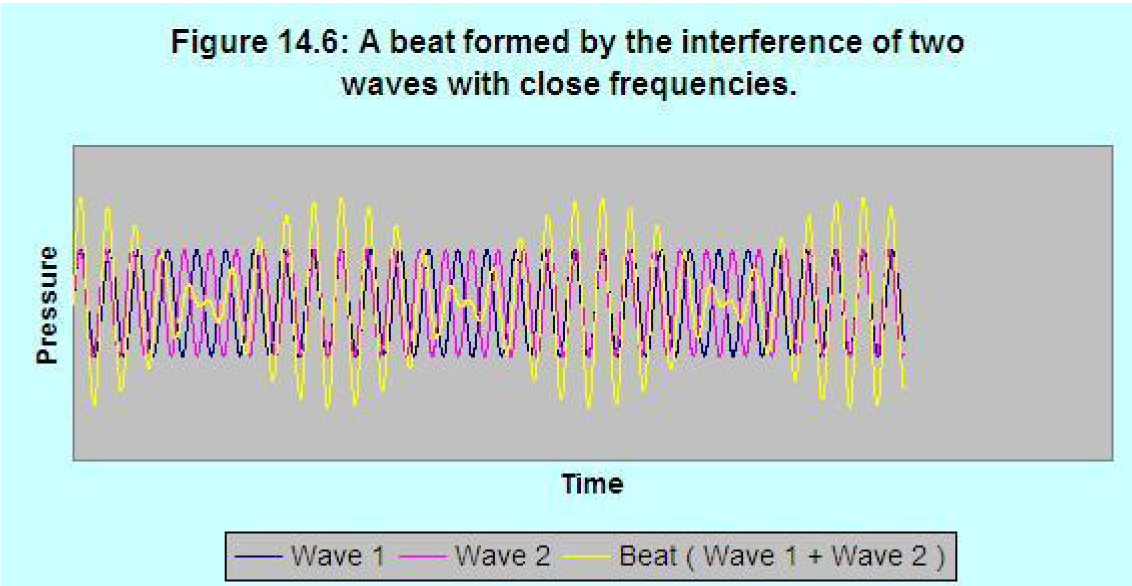

Figure 14.5

The part of the wave between two consecutive points of zero vibration constitutes one beat. The time taken for one beat is called the beat period (T_{beat}). The number of beats heard per second is called beat frequency (f_{beat}). The frequency of the beat is equal to the difference between the frequencies of the interfering waves. If the frequencies of the interfering waves are f_1 and f_2 , then the beat frequency is given as follows:

$$f_{beat} = | f_2 - f_1 |$$

The period of the beat is the inverse of the frequency of the beat.

$$T_{beat} = 1 / f_{beat}$$

Example: A beat is formed by the interference of two sound waves of frequencies *500* Hz and *505* Hz.

a) How many beats per second are heard?

Solution: $f_1 = 500$ Hz; $f_2 = 505$ Hz; $f_{beat} = ?$

$$f_{beat} = |f_2 - f_1| = |505 - 500| \text{ Hz} = 5 \text{ Hz}$$

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b) How long does each beat last?

Solution: $T_{beat} = ?$

$$T_{beat} = 1 / f_{beat} = 1 / 5 \text{ s} = 0.2 \text{ s}$$

Example: When a sound of frequency *650* Hz interferes with a sound of unknown frequency, *2* beats are heard per second. What are the two possible frequencies of the sound?

Solution: $f_{beat} = 2$ Hz; $f_1 = 650$ Hz; $f_2 = ?$

$$f_{beat} = |f_2 - f_1| = 2 \text{ Hz}$$

$$f_2 - 650 \text{ Hz} = \pm 2 \text{ Hz}$$

$$f_2 = (650 \pm 2) \text{ Hz} = 652 \text{ Hz or } 648 \text{ Hz}$$

14.9 Practice Quiz 14.2

Choose the best answer. Answers can be found at the back of the book.

1. Which of the following is not true about standing waves.
   A. One loop of a standing wave is equal to half of the wavelength of the wave in length.
   B. The speed of a wave in a string is proportional to the square root of the tension of the string.
   C. A standing wave is formed when an oncoming wave and a reflected wave interfere.
   D. A standing wave is a wave with zero vibrations at equal intervals of time.
   E. The points of zero vibration of a standing wave are called nodes.

2. Which of the following is correct about a beat?
   A. A beat is formed when two waves of close frequencies interfere.
   B. A beat is formed when two waves of the same frequency interfere.
   C. The period of a beat is equal to the difference between the periods of the interfering waves.
   D. A beat is a wave where points of zero vibration are separated by equal distances.
   E. The frequency of a beat is equal to the sum of the frequencies of the interfering waves.

3. An object of mass 13 kg is hanging from a string of length 1.2 m. The mass of the string is 0.012 kg. Calculate the speed of a wave excited on the string.
   A. 112.872 m / s
   B. 101.584 m / s
   C. 79.01 m / s
   D. 67.723 m / s
   E. 135.446 m / s

4. There is a tension of *70* N in a string of mass *0.012* kg and length *1.2* m. Calculate the frequency of a wave of wavelength *0.9* m excited in the string.
   A. *130.147* Hz
   B. *92.962* Hz
   C. *46.481* Hz
   D. *120.851* Hz
   E. *111.555* Hz

5. A standing wave formed in a *2* m string clamped at both ends has *4* loops. Calculate the wavelength of the wave.
   A. *0.6* m
   B. *1.3* m
   C. *1* m
   D. *0.9* m
   E. *0.8* m

6. A string of mass *0.14* kg and length *1.3* m is clamped at both ends. It is subjected to a tension of *150* N. Calculate the frequency of the 7^{th} harmonic standing wave formed in the string.
   A. *140.671* Hz
   B. *90.432* Hz
   C. *100.48* Hz
   D. *120.576* Hz
   E. *70.336* Hz

7. A sound wave resonance is formed in a pipe of length *2.5* m open at both ends. Calculate the wavelength of the 5^{th} harmonic.
   A. *1.2* m
   B. *0.6* m
   C. *1* m
   D. *0.7* m
   E. *0.9* m

8. A sound wave resonance is formed in a pipe of length *3* m open at both ends. Calculate the frequency of the 8^{th} harmonic. (The temperature is *20* °C.)
   A. *457.214* Hz
   B. *365.771* Hz
   C. *274.328* Hz
   D. *320.05* Hz
   E. *411.492* Hz

9. A sound wave resonance is formed in a pipe of length *4.5* m open at one end closed at the other. Calculate the wavelength of the *5* th harmonic.
   A. *2* m
   B. *1.8* m
   C. *2.6* m
   D. *2.8* m
   E. *1.6* m

10. A sound wave resonance is formed in a pipe of length *2.5* m open at one end closed at the other. Calculate the frequency of the *5* th harmonic. (The temperature is *25* °C.)
   A. *342.366* Hz
   B. *186.745* Hz
   C. *311.241* Hz
   D. *404.614* Hz
   E. *248.993* Hz

11. A beat is formed by the interference of a wave of frequency *107* Hz and a wave of frequency *101.5* Hz. Calculate the time taken for one beat.
   A. *0.005* s
   B. *208.5* s
   C. *0.182* s
   D. *0.001* s
   E. *5.5* s

12. A beat is formed by the interference of a wave of frequency *200* Hz and a wave of an unknown frequency. If a beat lasts *0.1* s, which of the following is a possible frequency for the unknown wave?
   A. *147* Hz
   B. *189* Hz
   C. *210* Hz
   D. *168* Hz
   E. *126* Hz

Answers to Practice Quizzes

Practice Quiz 1.1

C 2. A 3. A 4. C 5. D 6. C 7. D 8. D 9. D 10. A 11. C 12. D

Practice Quiz 1.2

1. D 2.B 3.B 4. E 5. E 6. B 7. B 8. B 9. C 10. E 11. C 12. C 13. C

Practice Quiz 2.1

1. B 2. A 3. D 4. B 5. E 6. B 7. B 8. C 9. A 10. B 11. C 12. C 13. C

Practice Quiz 2.2

1. B 2. E 3. D 4. C 5. A 6. E 7. C 8. B 9. D 10. C 11. D

Practice Quiz 3.1

1. A 2. A 3. B 4. D 5. D 6. D 7. A 8. B. 9. D 10. C 11. D 12. B 13. D 14. D 15. B

Practice Quiz 3.2

1. D 2. C 3. C 4. A 5. C 6. B 7. C 8. D 9. D 10. D

Practice Quiz 4.1

1. D 2. A 3. C 4. B 5. B 6. A 7. C 8. C 9. A 10. D 11. C

Practice Quiz 4.2

1. C 2. C 3. B 4. B 5. E 6. E 7. E 8. B 9. E 10. D

Practice Quiz 5.1

1. A 2. E 3. A 4. A 5. C 6. B 7. E 8. C 9. D 10. D 11. E

Practice Quiz 5.2

1. D 2. E 3. A 4. C 5. D 6. B 7. E 8. C 9. A 10. C 11. B

Practice Quiz 6.1

1. A 2. A 3. A 4. D 5. A 6. C 7. A 8. D 9. C 10. E

Practice Quiz 6.2

1. D 2. C 3. E 4. E 5. C 6. A 7. D 8. A 9. E 10. B

Practice Quiz 7.1

1. E 2. A 3. D 4. C 5. D 6. D 7. D 8. B 9. B 10. C 11. E 12. A 13. D 14. E

Practice Quiz 7.2

1. A 2. B 3. E 4. D 5. C 6.B 7. C 8. A 9. D 10. D

Practice Quiz 8.1

1. B 2. A 3. B 4. D 5.A 6. A 7. B 8. B 9. E 10. C

Practice Quiz 8.2

1. B 2. D 3. C 4. B 5. C 6. A 7. E 8. A 9. B 10. D

Practice Quiz 9.1

1. B 2. A 3. C 4. B 5. B 6. A 7. B 8. A 9. B 10. B

Practice Quiz 9.2

1. E 2. B 3. A 4. E 5. A 6. A 7. C 8. B 9. D 10. B

Practice Quiz 10.1

1. B 2. C 3. A 4. C 5. C 6. A 7. E 8. E 9. C 10. B

Practice Quiz 10.2

1. A 2. D 3. A 4. D 5. B 6. C 7. C 8. B 9. C 10. B

Practice Quiz 11.1

1. D 2. C 3. E 4. C 5. E 6. E 7. E 8. A 9. C 10. E

Practice Quiz 11.2

1. B 2. E 3. E 4. E 5. E 6. C 7. B 8. A

Practice Quiz 12.1

1. D 2. C 3. E 4. A 5. B 6. D 7. E 8. C 9. C 10. A

Practice Quiz 12.2

1. D 2. A 3. E 4. E 5. D 6. D

Practice Quiz 13.1

1. B 2. E 3. B 4. A 5. E 6. A 7. B 8. E 9. E 10. B

Practice Quiz 13.2

1. E 2. D 3. C 4. B 5. B 6. C 7. B 8. D 9. E 10. C

Practice Quiz 14.1

1. B 2. A 3. C 4. C 5. E 6. C 7. A 8. A 9. E 10. B

Practice Quiz 14.2

1. D 2. A 3. A 4. B 5. C 6. C 7. C 8. A 9. A 10. C 11. C 12. C